Lateral stress caused by horizontal and vertical surcharge strip loads on a cross-anisotropic backfill

Cheng-Der Wang*+,†

Department of Civil and Disaster Prevention Engineering, National United University,
No. 1, Lien-Da, Kung-Ching-Li, Miao-Li, 360, Taiwan, ROC

SUMMARY

This study derives analytical solutions for estimating the lateral stress caused by horizontal and vertical surcharge strip loads resting on a cross-anisotropic backfill. The following loading types are employed in this work: point load, line load, uniform strip load, upward linear-varying strip load, upward nonlinear-varying strip load, downward linear-varying strip load and downward nonlinear-varying strip load. The cross-anisotropic planes are assumed to be parallel to the horizontal surface of the backfill. The solutions proposed herein have never been mentioned in previous literature, but can be derived by integrating the point load solution in a Cartesian co-ordinate system for a cross-anisotropic medium. The calculations by the presented solutions are quick and accurate since they are concise and systematized. Additionally, the proposed calculations demonstrate that the type and degree of material anisotropy and the horizontal/vertical loading types decisively influence the lateral stress. This investigation presents examples of the proposed horizontal and vertical strip loads acting on the surface of the isotropic and cross-anisotropic backfills to elucidate their effects on the stress. The analytical results reveal that the stress distributions accounting for soil anisotropy and loading types are quite different from those computed from the available isotropic solutions. Restated, the derived solutions, as well as realistically simulating the actual surcharge loading circumstances, provide a good reference for the design of retaining structures for the backfill materials are cross-anisotropic. Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: analytical solutions; lateral stress; horizontal and vertical surcharge strip loads; cross-anisotropic backfill; point load; Cartesian co-ordinate system

INTRODUCTION

In engineering, the retaining structure is frequently subjected to surcharge loads on its backfill. The superimposed surface loads could be any loading comprising point loads, such as truck wheels, or distributed loads, such as foundations of adjacent buildings, highway pavements and railroad tracks. When the surcharge loads on the backfill close enough to the wall, additional lateral pressure could be generated [1–3]. Thus, the magnitude and distribution of lateral stress

*Correspondence to: Cheng-Der Wang, Department of Civil and Disaster Prevention Engineering, National United University, No. 1, Lien-Da, Kung-Ching-Li, Miao-Li, 360, Taiwan, ROC.
†E-mail: cdwang@nuu.edu.tw, cdwang0720@pchome.com.tw
‡Associate Professor.

Received 14 August 2004
Revised 8 May 2005
Accepted 15 June 2005

Copyright © 2005 John Wiley & Sons, Ltd.
on the retaining wall caused by loads superimposed upon the surface of the backfill are of interest in geotechnical engineering [4–6]. Previously, the backfill material typically has been assumed to be homogeneous, linearly elastic and isotropically continuous. The classical isotropic elastic solutions for loads applied to the surface of a half-space, readily available in texts such as Poulos and Davis [7] and Clayton et al. [8], are applied to predict the lateral stress increase on a wall at any depth. However, this study recognizes that soils are not generally isotropic materials. Normally, most soils exhibit an axial or transverse isotropic structure often referred to as cross-anisotropic, with identical properties in all directions within the horizontal plane, which are different from the properties in the vertical direction, which is the direction of deposition. Hence, better results can only be yielded by considering anisotropic deformability [9–16]. This work derived analytical solutions for lateral stress resulting from a horizontal/vertical line load, and various horizontal/vertical strip loads acting on the cross-anisotropic backfill.

The theory of elasticity method is well known to be applicable in computing the lateral pressure profile against the wall from the surface surcharge loading [17]. The general validity of using the Boussinesq’s solution [18] for surcharges was established in several publications, including Spangler [1, 19,20], Spangler and Handy [21], Spangler and Mickle [4], Rehnman and Broms [22] and others [17]. Engineers at the Iowa Engineering Experiment Station performed a series of experiments to determine the lateral pressure on a wall due to concentrated loads applied at the backfill surface and the uniformly distributed line or strip loads parallel to the wall. These experiments indicated that the surface loads produced lateral pressures which were closely related to the pressures calculated by the Boussinesq’s solution [18] in a semi-infinite elastic medium, and they also provided a basis for further study of the effect of loads applied at the backfill surface [4]. Nevertheless, the above-mentioned experiments showed that the measured lateral pressure was about twice that computed by the Boussinesq’s solution [18] within the half-space with Poisson’s ratio = 0.5. Mindlin [23] discussed Spangler’s work [19], and suggested that the doubling of the elastic solution [18] could be explained by applying an image-based approach to a perfectly smooth rigid retaining wall. This configuration might cause a mirror load to be placed symmetrically in front of the wall. Mindlin [23] demonstrated that the horizontal displacements at the rigid wall would be zero, enabling his method to be invoked to predict the lateral stress [8]. Hence, the resulting stress is about twice that obtained using the Boussinesq’s solution [18]. Rehnman and Broms [24] revealed that the lateral pressure from point loads when the soil behind the wall was dense was much lower than when the soil was loose, and that the gravelly backfill generated higher pressure than finer-grained materials. This observation indicated that both the soil state and Poisson’s ratio were significant parameters for the lateral stress. Misra [25] theoretically measured the variation of lateral pressure distribution resulting from different types of granular backfill subjected to various levels of contact pressure on its surface using the equation, $E/G > 2(1 + v)$. Jarquio [26] derived a simplified expression for determining the centroid of total lateral surcharge pressure, and the point of maximum unit lateral pressure for both yielding and unyielding retaining wall structures. The author is not aware of any proposed analyses of lateral stress for a cross-anisotropic backfill caused by surcharge loads. In deriving the proposed solutions, the backfill material is assumed to be a homogeneous, linearly elastic and cross-anisotropic continuum. The retaining wall is vertical with horizontal backfill, that is, the cross-anisotropic planes are parallel to the boundary plane. Additionally, this investigation applies two simplifying assumptions: (1) the wall does not move, and (2) the wall is perfectly smooth (no friction between the wall and the soil). Under these
Circumstances, the induced lateral stress on the wall would be the same as the induced horizontal stress in an elastic half-space by two loads of equal magnitude [27]. Hence, seven cases of horizontal and vertical loading are explored as follows:

- **Case A**: horizontal/vertical point load, \( P/Q \) (force).
- **Case B**: horizontal/vertical line load, \( P/L = Q/L \) (force per unit length).
- **Case C**: horizontal/vertical uniform strip load, \( P_s/Q_s \) (force per unit area).
- **Case D**: horizontal/vertical upward linear-varying strip load, \( P_s/Q_s \) (maximum force per unit area).
- **Case E**: horizontal/vertical upward nonlinear-varying strip load, \( P_s/Q_s \) (maximum force per unit area).
- **Case F**: horizontal/vertical downward linear-varying strip load, \( P_s/Q_s \) (maximum force per unit area).
- **Case G**: horizontal/vertical downward nonlinear-varying strip load, \( P_s/Q_s \) (maximum force per unit area).

The exact solutions proposed in this article can be directly obtained by integrating the point load solutions in a Cartesian coordinate system for a cross-anisotropic medium [28]. The derived solutions are clear and concise, and show that the type and degree of material anisotropy, and different horizontal/vertical loading types deeply influence the induced lateral stress. Two examples are presented at the end of this study to illustrate the ratio of anisotropic to isotropic lateral stress in the isotropic/cross-anisotropic backfills owing to a horizontal/vertical uniform strip load, a horizontal/vertical upward linear-varying strip load and a horizontal/vertical upward nonlinear-varying strip load.

**CASE A: LATERAL STRESS CAUSED BY A HORIZONTAL/VERTICAL POINT LOAD**

In this work, the solutions of lateral stress caused by horizontal/vertical surcharge strip loads on a cross-anisotropic backfill are obtained by integrating the point load solution in a Cartesian coordinate system [28]. The cross-anisotropic planes are assumed to be parallel to the horizontal ground surface. Figure 1 depicts the proposed approaches for solving the lateral stress subjected to a horizontal point load \( P \), and a vertical point load \( Q \), which as the form of body forces

![Figure 1. Superposition approach to the point loading half-space problem.](image-url)
[29, point load solutions presented in terms of the cylindrical co-ordinate system]. Figure 1 demonstrates that a half-space comprises two full-spaces, one with a horizontal/vertical point load in its interior \((0,0,h)\), and the other with opposite traction of the first full-space along \(z = 0\). The traction in the first full-space along \(z = 0\) results from the point load. The solutions for the half-space are thus derived by superposing the solutions of the two full-spaces. Restated, the solutions can be obtained from the governing equations for a full-space (including the general solutions (I) and homogeneous solutions (II)) by adhering to the traction-free boundary conditions on the half-space surface. If \(h = 0\), then the horizontal/vertical point load is applied to the surface. Moreover, since the surcharge is usually applied a certain distance from the wall, the distance at which a load is applied should be considered [5]. Therefore, this work applied the horizontal/vertical point load, \(P/Q\), at a horizontal distance of \(a\) from the retaining wall with height \(H(x = a, y = 0, z = 0)\), as revealed in Figure 2(a). Thus, the solutions for lateral stress on a cross-anisotropic backfill can be directly integrated from the point load solutions [28]. Figures 2(b)–(g), respectively, display solutions for lateral stress owing to a horizontal/vertical line load, a horizontal/vertical uniform strip load, a horizontal/vertical upward linear-varying strip load, a horizontal/vertical upward nonlinear-varying strip load, a horizontal/vertical downward linear-varying strip load and a horizontal/vertical downward nonlinear-varying strip load. The analytical solution for horizontal stress \(\sigma_{x}^p\) in the Cartesian co-ordinate system subjected to a horizontal point load \(P\) and a vertical point load \(Q\) on the surface of a cross-anisotropic medium can be recast as follows:

\[
\sigma_{x}^p = \frac{P}{2\pi} \left\{ \frac{k(m_2u_1 - m_1u_2)}{m_1m_2(u_1 - u_2)} \left[ A_{44}(u_1^2p_{31} - u_2^2p_{32}) ight] - 2A_{66} \left( \frac{u_1}{m_1 + u_1} p_{321} - \frac{u_2}{m_2 + u_2} p_{322} \right) \right\} + \frac{Q}{2\pi} k(m_2u_1 - m_1u_2) \left\{ A_{44}(u_1p_{31} - u_2p_{32}) - 2A_{66} \left( \frac{1}{m_1 + u_1} (p_{31} - p_{341}) - \frac{1}{m_2 + u_2} (p_{32} - p_{342}) \right) \right\}
\]

(1)

However, for a perfectly smooth rigid retaining wall, the lateral stress \(\sigma_{x}^p\) is assumed to be twice as great as that calculated from the cross-anisotropic point load solution (Equation (1)). Hence, Equation (1) can be further rewritten as

\[
\sigma_{x}^p = 2* \frac{P}{2\pi} \left\{ \frac{k(m_2u_1 - m_1u_2)}{m_1m_2(u_1 - u_2)} \left[ A_{44}(u_1^2p_{31} - u_2^2p_{32}) ight] - 2A_{66} \left( \frac{u_1}{m_1 + u_1} p_{321} - \frac{u_2}{m_2 + u_2} p_{322} \right) \right\} + \frac{Q}{2\pi} k(m_2u_1 - m_1u_2) \left\{ A_{44}(u_1p_{31} - u_2p_{32}) - 2A_{66} \left( \frac{1}{m_1 + u_1} (p_{31} - p_{341}) - \frac{1}{m_2 + u_2} (p_{32} - p_{342}) \right) \right\}
\]

(2)
Figure 2. Lateral stress caused by various types of horizontal/vertical surcharge loads on a cross-anisotropic backfill: (a) point load case; (b) line load case; (c) uniform strip load case; (d) upward linear-varying strip load case; (e) upward nonlinear-varying strip load case; (f) downward linear-varying strip load case; and (g) downward nonlinear-varying strip load case.
The generalized Hooke’s law for the cross-anisotropic medium in a Cartesian co-ordinate system can be adopted to express the constitutive equations employed herein as

\[
\sigma_{xx} = A_{11}e_{xx} + (A_{11} - 2A_{66})e_{yy} + A_{13}e_{zz}
\]

\[
\sigma_{yy} = (A_{11} - 2A_{66})e_{xx} + A_{11}e_{yy} + A_{13}e_{zz}
\]

\[
\sigma_{zz} = A_{13}(e_{xx} + e_{yy}) + A_{33}e_{zz}
\]

\[
\tau_{xy} = A_{66}\gamma_{xy}
\]

\[
\tau_{yz} = A_{44}\gamma_{yz}
\]

\[
\tau_{xz} = A_{44}\gamma_{xz}
\]

\( A_{ij} \) \((i,j = 1-6)\) denote the elastic moduli or elasticity constants of the medium [29, 30]. For a cross-anisotropic material, the five engineering elastic constants, \( E, E', v, v' \), and \( G' \) are defined as [31]

1. \( E \) represents the Young’s modulus in the horizontal direction.
2. \( E' \) represents the Young’s modulus in the vertical direction.
3. \( v \) represents the Poisson’s ratio for the effect of horizontal stress on complementary horizontal strain.
4. \( v' \) represents the Poisson’s ratio for the effect of vertical stress on horizontal strain.
5. \( G' \) represents the shear modulus in the vertical plane.

Hence, \( A_{ij} \) \((i,j = 1-6)\) can be expressed in terms of these elastic constants as

\[
A_{11} = \frac{E(1 - (E/E')v'^2)}{(1 + v)(1 - v - (2E/E')v'^2)}, \quad A_{13} = \frac{Ev'}{1 - v - (2E/E')v'^2},
\]

\[
A_{33} = \frac{E'(1 - v)}{1 - v - (2E/E')v'^2}, \quad A_{44} = G', \quad A_{66} = \frac{E}{2(1 + v)}
\]

Thermodynamic constraints stipulate that the strain energy of an elastic material should always be positive. Therefore, the theoretical bounding values of the relevant elastic parameters are described as [11, 32]:

\[
E, E', G, G' > 0
\]

\[
-1 < v < 1, \quad -\sqrt{\frac{E' 1 - v}{2}} < v' < \sqrt{\frac{E' 1 - v}{2}}
\]
• $u_3 = \sqrt{A_{66}/A_{44}}$, $u_1$ and $u_2$ denote the roots of the following characteristic equation:

$$u^4 - su^2 + t = 0$$

where

$$s = \frac{A_{11}A_{33} - A_{13}(A_{13} + 2A_{44})}{A_{33}A_{44}}, \quad t = \frac{A_{11}}{A_{33}}$$

The characteristic roots, $u_1$ and $u_2$, can be categorized into three cases as follows:

Case 1: $u_{1,2} = \pm \sqrt{\{s \pm \sqrt{(s^2 - 4t)}\}}$ has two real distinct roots when $s^2 - 4t > 0$.

Case 2: $u_{1,2} = \pm \sqrt{s/2}, \pm \sqrt{s/2}$ has double equal real roots when $s^2 - 4t = 0$ (i.e. complete isotropy).

Case 3: $u_1 = \frac{1}{2} \sqrt{(s + 2\sqrt{i}) - i \frac{1}{2} \sqrt{(-s + 2\sqrt{i})}} = \gamma - i\delta$, $u_2 = \gamma + i\delta$ has two complex conjugate roots (where $\gamma \neq 0$) when $s^2 - 4t < 0$.

Using engineering elastic constants, the following criterion [33, 34] also can identify the root type of Equation (12).

$$\left(\frac{G}{G'}\right)^2 (1 + v) - \left(\frac{E}{E'}\right) \left[1 - v + \left(\frac{E}{E'}\right) v' - 2 \left(\frac{E}{E'}\right) v^2\right]$$

$$\begin{cases}
> 0 & \text{for Case 1} \\
= 0 & \text{for Case 2} \\
< 0 & \text{for Case 3}
\end{cases}$$

• $k = \frac{(A_{13} + A_{44})}{A_{33}A_{44}(u_1^2 - u_2^2)}$, $m_j = \frac{(A_{13} + A_{44})u_j}{A_{33}u_j^2 - A_{44}} = \frac{A_{11} - A_{44}u_j^2}{(A_{13} + A_{44})u_j} (j = 1, 2)$, $p_{s1i} = \frac{x + a}{R_i}$

$$m_i = \frac{3(x + a)}{R_i(R_i + z_i)} + \frac{(x + a)^3(3R_i + z_i)}{R_i^3(R_i + z_i)^3}, \quad p_{33i} = \frac{z_i}{R_i}$$

$$p_{33i} = \frac{1}{R_i(R_i + z_i)} - \frac{(x + a)^3(2R_i + z_i)}{R_i^3(R_i + z_i)^2}, \quad R_i = \sqrt{(x + a)^2 + y^2 + z_i^2}, \quad z_i = u_iz \quad (i = 1, 2, 3)$$

Notably, for a backfill with double equal real roots (Case 2), the exact solutions for the lateral stress can be computed from Equation (2) by making $u_2$ approach $u_1$, and using the L’Hôpital rule. However, when $u_1 = u_2 = 1$, Equation (2) is only valid for an isotropic backfill in a limiting sense.

**CASE B: LATERAL STRESS CAUSED BY A HORIZONTAL/ VERTICAL LINE LOAD**

The theory of elasticity can be adopted to determine the lateral stress on retaining structures caused by various types of surcharge loading, such as a line load or a strip load. Hence, if a parallel line load such as a footing is very long, it may extend from any point on the wall to $-\infty$ and $\infty$. In the case of a horizontal/vertical line load with an intensity $P_i/Q_i$ (force per unit length, as illustrated in Figure 2(b)) acting at a horizontal distance $a$ from a wall with height $H$ on a cross-anisotropic backfill, the lateral stress $\sigma_{li}$ at any depth $z$ can be calculated by integrating Equation (2) using the standard integration formulae [35]. In other words, the
analytical solution $\sigma_h^l$ can be obtained by integrating Equation (2) in the $y$ direction from $-\infty$ to $\infty$ as follows:

$$\sigma_h^l = \int_{-\infty}^{\infty} \sigma_h^p \, dy$$

(14)

The solution for lateral stress due to a horizontal/vertical line load resting on a cross-anisotropic backfill can be written as

$$\sigma_h^l = 2 \cdot P^i \cdot n \left[ \frac{u_1^2(x+a)}{(x+a)^2 + (u_1z)^2} - \frac{u_2^2(x+a)}{(x+a)^2 + (u_2z)^2} \right]$$

$$+ 2 \cdot Q^j \cdot n \cdot m_1 \cdot m_2 \left[ \frac{u_1^2z}{(x+a)^2 + (u_1z)^2} - \frac{u_2^2z}{(x+a)^2 + (u_2z)^2} \right]$$

(15)

where $n = A_4 k (m_2 u_1 - m_1 u_2)/\pi m_1 m_2 (u_1 - u_2)$.

**CASE C: LATERAL STRESS CAUSED BY A HORIZONTAL/VERTICAL UNIFORM STRIP LOAD**

Retaining wall structures supporting continuous wall footing, highway and railroad loading are practical examples applying the strip load surcharge [26]. Figure 2(c) in this section depicts a horizontal/vertical uniform strip load with an intensity $P^i/Q^j$ (force per unit area) and width $b$, at a horizontal distance $a$ from a wall with height $H$ on a cross-anisotropic backfill. To solve the lateral stress induced by this load, the complete solution can be obtained by integrating the line load solution ($\sigma_h^l$), Equation (15) in the $x$ direction between the limits 0 and $b$ to describe a surcharge load uniformly distributed on a strip, as follows:

$$\sigma_h^u = \int_{0}^{b} \sigma_h^l \, dx$$

(16)

where $\sigma_h^u$ denotes the lateral stress caused by a horizontal/vertical uniform strip load. Upon integration, the explicit solution can be given by

$$\sigma_h^u = 2 \cdot P^i \cdot n \cdot (u_1^2 I_{c1} - u_2^2 I_{c2}) + 2 \cdot Q^j \cdot n \cdot m_1 \cdot m_2 \cdot (u_1 I_{d1} - u_2 I_{d2})$$

(17)

where

$$I_{cj} = \ln \left| \frac{\sqrt{(a/u_2 z) + (b/u_1 z)^2} + 1}{\sqrt{(a/u_2 z)^2 + 1}} \right|, \quad I_{dj} = \tan^{-1} \left( \frac{a}{u_j z} + \frac{b}{u_j z} \right) - \tan^{-1} \left( \frac{a}{u_j z} \right), \quad (j = 1, 2)$$

Equation (17) shows that $I_{cj}$ and $I_{dj}$ are both functions of $a/u_1 z$ and $b/u_1 z$ ($j = 1, 2$). Therefore, if the five engineering elastic constants $E, E', v, v', G'$ are given, then the characteristic root $u_j$ ($j = 1, 2$) can be computed using Equations (9) and (12). Figures 3 and 4 plot the calculation charts for $I_{cj}$ and $I_{dj}$, with variable nondimensional factors, $a/u_1 z$ and $b/u_1 z$, in which $a/u_1 z$ is in the range 0.01–10, and $b/u_1 z$ is in the range 0.1–10. These figures can be utilized when computers or calculators are unavailable, but are only appropriate for the root type of the characteristic
equation (Equation (13)) belonging to Case 1 (i.e. where $u_1$ and $u_2$ are two real distinct roots).

Hence, if the root type belongs to Case 2 or Case 3, the presented calculation charts (Figures 3 and 4) cannot be employed to calculate the lateral stress due to a horizontal/vertical uniform strip load.

**CASE D: LATERAL STRESS CAUSED BY A HORIZONTAL/VERTICAL UPWARD LINEAR-VARYING STRIP LOAD**

For an induced horizontal/vertical load with a nonuniform distribution, a horizontal/vertical upward linear-varying strip load is first used. Figure 2(d) indicates that the load is upward linearly varied in the $x$ direction from 0 (at $x = a$) to the maximum magnitude $P_s$ (at $x = a + b$),

![Figure 3. Calculation chart for $I_{cj}$.](image)
on a strip with width \( b \). The lateral stress \( \sigma_{h}^{\text{lin}} \) is given by directly integrating the line load solution (\( \sigma_{h} \), Equation (15)) as follows:

\[
\sigma_{h}^{\text{lin}} = \int_{0}^{b} \left( \frac{x}{b} \right) \sigma_{h}^{1} \, dx
\]

\[
= 2 * P_{s} * n * \left\{ u_{1}^{2} \left[ 1 - \left( \frac{u_{1} z}{b} \right) I_{d1} - \left( \frac{a}{b} \right) I_{c1} \right] - u_{2}^{2} \left[ 1 - \left( \frac{u_{2} z}{b} \right) I_{d2} - \left( \frac{a}{b} \right) I_{c2} \right] \right\}

- 2 * Q_{s} * n * m_{1} * m_{2} * \left\{ u_{1} \left[ \left( \frac{a}{b} \right) I_{d1} - \left( \frac{u_{1} z}{b} \right) I_{c1} \right] - u_{2} \left[ \left( \frac{a}{b} \right) I_{d2} - \left( \frac{u_{2} z}{b} \right) I_{c2} \right] \right\} \tag{18}
\]

where \( P_{s} / Q_{s} \) represents the maximum horizontal/vertical upward linear-varying strip load (force per unit area). If computers or calculators are unavailable, calculation charts in Figures 3 and 4 can also be utilized to estimate the desired stress if the root type of a cross-anisotropic backfill belongs to Case 1.

Copyright © 2005 John Wiley & Sons, Ltd.

CASE E: LATERAL STRESS CAUSED BY A HORIZONTAL/VERTICAL UPWARD NONLINEAR-VARYING STRIP LOAD

Frequently, applied loads do not vary linearly varying but can be more realistically depicted as nonlinearly varying distributed [36]. A nonlinear load upwardly distributed as a quadratic variation on a strip (Figure 2(e)) is considered to simulate this loading case. This induced lateral stress, \( s_{\text{u \ non}} \), by a horizontal/vertical upward nonlinear-varying strip load, can be obtained by integrating the line load solution (\( s_{\text{l \ h}} \), Equation (15)) as follows:

\[
\sigma_{\text{h \ non}} = \int_{0}^{b} \left( \frac{x}{b} \right)^{2} \sigma_{\text{l \ h}} \, dx
\]

\[
= 2 \times P_{s} \times n \times \left[ \frac{1}{2} - \frac{a}{b} \right] + 2 \left( \frac{a}{b} \right) \left( \frac{u_{1} z}{b} \right) I_{d1} + \left[ \left( \frac{a^{2}}{b^{2}} \right) - \left( \frac{u_{1} z^{2}}{b^{2}} \right) \right] I_{c1}
\]

\[
- u_{2} \left[ \frac{1}{2} - \frac{a}{b} \right] + 2 \left( \frac{a}{b} \right) \left( \frac{u_{2} z}{b} \right) I_{d2} + \left[ \left( \frac{a^{2}}{b^{2}} \right) - \left( \frac{u_{2} z^{2}}{b^{2}} \right) \right] I_{c2}
\]

\[
+ 2 \times Q_{s} \times n \times m_{1} \times m_{2} \times \left[ \frac{1}{2} - \frac{a}{b} \right] + \left[ \left( \frac{a^{2}}{b^{2}} \right) - \left( \frac{u_{1} z^{2}}{b^{2}} \right) \right] I_{d1} - 2 \left( \frac{a}{b} \right) \left( \frac{u_{1} z}{b} \right) I_{c1}
\]

\[
- u_{2} \left[ \frac{1}{2} - \frac{a}{b} \right] + \left[ \left( \frac{a^{2}}{b^{2}} \right) - \left( \frac{u_{2} z^{2}}{b^{2}} \right) \right] I_{d2} - 2 \left( \frac{a}{b} \right) \left( \frac{u_{2} z}{b} \right) I_{c2}
\]

where \( P_{s} / Q_{s} \) denotes the maximum horizontal/vertical upward nonlinear-varying strip load (force per unit area).

CASE F: LATERAL STRESS CAUSED BY A HORIZONTAL/VERTICAL DOWNWARD LINEAR-VARYING STRIP LOAD

Figures 2(f) and (g) illustrate two different cases for the lateral stress owing to a horizontal/vertical downward linear-varying strip load and a horizontal/vertical downward nonlinear-varying strip load, respectively. In Figure 2(f), the load varies linearly downward in the \( x \) direction from the maximum magnitude \( P_{s} \) (at \( x = a \)) to 0 (at \( x = a + b \)), on a strip with width \( b \). The lateral stress solution, \( \sigma_{\text{h \ lin}} \), can be obtained by combining loading Case C (Equation (17)) and Case D (Equation (18)). Hence, \( \sigma_{\text{h \ lin}} \) can be expressed simply as

\[
\sigma_{\text{h \ lin}} = \sigma_{\text{h \ u}} \text{ (Equation (17))} - \sigma_{\text{h \ u \ lin}} \text{ (Equation (18))}
\]

CASE G: LATERAL STRESS CAUSED BY A HORIZONTAL/VERTICAL DOWNWARD NONLINEAR-VARYING STRIP LOAD

Finally, this section considers a horizontal/vertical downward nonlinear-varying strip load applied to a cross-anisotropic backfill, as displayed in Figure 2(g). The explicit expression for \( \sigma_{\text{h \ non}} \) is a combination of loading Case C (Equation (17)), Case D (Equation (18)) and Case E
(Equation (19)). Therefore, \( \sigma_{hd}^{\text{non}} \) is written as
\[
\sigma_{hd}^{\text{non}} = \sigma_h^d \text{ (Equation (17))} - 2 \cdot \sigma_{h}^{u-\text{ln}} \text{ (Equation (18))} + \sigma_{h}^{u-\text{non}} \text{ (Equation (19))} \tag{21}
\]

Table I summarizes the proposed solutions, listing loading Cases C–G with respect to the induced lateral stress solutions. Based on Table I, the lateral stress caused by any conceivable surcharge strip load can also be analysed by superposing the presented loading cases. Figure 5 shows a flow chart for computing the lateral stress by the presented loading cases using the calculation charts, \( I_{cj} \) and \( I_{dj} \) (\( j = 1, 2 \)). Although the two charts are supplied for the five engineering elastic constants of a cross-anisotropic backfill belonging to Case 1, they provide an alternative tool for calculating the induced lateral stress quickly and accurately.

ILLUSTRATIVE EXAMPLES

This section describes a parametric study which was undertaken to confirm the derived solutions and to elucidate the effect of the type and degree of material anisotropy, and the loading type, on the lateral stress. For typical ranges of cross-anisotropic parameters, Gazetas [14] summarized several experimental data regarding deformational cross-anisotropy of clays and sands, and concluded that the ratio \( E/E' \) ranged from 0.6 to 4 for clays and was as low as 0.2 for sands. Hence, the influence of the degree of anisotropy, determined using the ratios \( E/E', v/v', \) and \( G/G' \), on the stress is examined. The backfill materials utilized herein are hypothetical isotropic (Soil 1 with \( E/E' = v/v' = G/G' = 1 \)) and cross-anisotropic sands (Soil 2 with \( E/E' = 0.2, v/v' = G/G' = 1 \); Soil 3 with \( E/E' = 2, v/v' = G/G' = 1 \)). Table II lists the elastic properties and root type of these materials.

Using Equations (2), (15), (17)–(21), a FORTRAN program was written to calculate the induced lateral stress under a point load, a line load and various strip loads. Two examples are illustrated to investigate the results of stress caused by horizontal and vertical strip loads for Soils 2 and 3, respectively.

Figures 6(a)–(c) show the ratio of anisotropy to isotropic lateral stress (Soil 2/Soil 1) for the backfills subjected to various horizontal strip loads, as follows. Figure 6(a) illustrates a horizontal uniform strip load, Figure 6(b) depicts a horizontal upward linear-varying strip load, and Figure 6(c) displays a horizontal upward nonlinear-varying strip load. In these graphs, the nondimensional factor \( a/b \) (where \( a \) denotes the load applied at a horizontal distance from the retaining wall, and \( b \) represents the width of the strip load, as shown in Figures 2(c)–(g)), for \( z/b = 0, 0.1, 0.5, 1, \) and 2 (where \( z \) is the vertical distance from point to load). To verify the accuracy of the proposed solutions, these results must be compared with the available isotropic solutions given in other investigations such as Poulos and Davis [7] and Clayton et al. [8]. These graphs demonstrate that the ratio of anisotropic to isotropic lateral stress induced by a horizontal uniform strip load (Figure 6(a)), a horizontal upward linear-varying strip load (Figure 6(b)) and a horizontal upward nonlinear-varying strip load (Figure 6(c)) approaches 0.9506, with \( a/b \) increasing from 0 to 10. The computed ratio for Soil 2/Soil 1 is nearly less than 1, except for the \( z/b = 1 \) case within \( a/b = 1 \), and \( z/b = 2 \) case within \( a/b = 2 \), due to each horizontal strip load.

Figures 7(a)–(c) plot the ratio of anisotropic to isotropic lateral stress (Soil 2/Soil 1) for the backfills subjected to the vertical strip loads. These figures reveal that the ratio of anisotropic to isotropic lateral stress induced by a vertical uniform strip load (Figure 7(a)), a vertical upward
Table I. Solutions of induced lateral stress caused by horizontal/vertical strip loading Cases C–G.

<table>
<thead>
<tr>
<th>Type of loading</th>
<th>Solutions</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case C</td>
<td>( \sigma_h^u = 2 \cdot P \cdot n \cdot (u_1^2 I_1 - u_2^2 I_2) + 2 \cdot Q \cdot n \cdot m_1 \cdot m_2 \cdot (u_1 I_1 - u_2 I_2) )</td>
<td>Figure 2(c), Equation (17)</td>
</tr>
<tr>
<td>Case D</td>
<td>( \sigma_h^{u,\text{lin}} = 2 \cdot P \cdot n \cdot \left{ u_1^2 \left[ 1 - \left( \frac{u_1 z}{b} \right) I_1 \right] - u_2^2 \left[ 1 - \left( \frac{u_2 z}{b} \right) I_2 \right] \right} - 2 \cdot Q \cdot n \cdot m_1 \cdot m_2 \cdot \left{ u_1 \left[ \frac{u_1 z}{b} I_1 \right] - u_2 \left[ \frac{u_2 z}{b} I_2 \right] \right} )</td>
<td>Figure 2(d), Equation (18)</td>
</tr>
<tr>
<td>Case E</td>
<td>( \sigma_h^{u,\text{non}} = 2 \cdot P \cdot n \cdot \left{ \frac{u_1^2}{2} - \left( \frac{a}{b} \right) + 2 \left( \frac{a}{b} \right)^2 \left[ \frac{u_1 z}{b} \right] I_1 \right} - \frac{u_2^2}{2} - \left( \frac{a}{b} \right) + 2 \left( \frac{a}{b} \right)^2 \left[ \frac{u_2 z}{b} \right] I_2 \right} + 2 \cdot Q \cdot n \cdot m_1 \cdot m_2 \cdot \left{ \frac{u_1}{2} \left[ \frac{u_1 z}{b} \right] I_1 - 2 \left( \frac{a}{b} \right) \left[ \frac{u_1 z}{b} \right] I_1 \right} )</td>
<td>Figure 2(e), Equation (19)</td>
</tr>
<tr>
<td>Case F</td>
<td>( \sigma_h^{d,\text{lin}} = \sigma_h^u ) (Equation (17)) - ( \sigma_h^{u,\text{lin}} ) (Equation (18))</td>
<td>Figure 2(f), Equation (20)</td>
</tr>
<tr>
<td>Case G</td>
<td>( \sigma_h^{d,\text{non}} = \sigma_h^u ) (Equation (17)) - 2 ( \sigma_h^{u,\text{lin}} ) (Equation (18)) + ( \sigma_h^{u,\text{non}} ) (Equation (19))</td>
<td>Figure 2(g), Equation (21)</td>
</tr>
</tbody>
</table>

where \( P/Q \) (maximum force per unit area) denotes the horizontal/vertical strip load; \( n \) denotes the load applied at a horizontal distance from the retaining wall; \( b \) denotes the width of the strip load; \( z \) denotes the vertical distance from point to load:

\[
\begin{align*}
 n &= \frac{A_{4k} (m_2 u_1 - m_1 u_2)}{\pi m_2 (u_1 - u_2)}, & k &= \frac{(A_{13} + A_{44})}{A_{33} A_{44} (u_1 - u_2)}, & m_y &= \frac{(A_{13} + A_{44}) u_y}{A_{33} u_y^2 - A_{44}}, & A_{11} &= A_{44} u_y^2 \\
 I_{ij} &= \ln \left| \sqrt{\left( \frac{a}{u_j} + \frac{b}{u_j z} \right)^2 + 1} \right| \sqrt{\left( \frac{a}{u_j z} \right)^2 + 1} \\
 I_{ij} &= \tan^{-1} \left( \frac{a}{u_j z} + \frac{b}{u_j z} \right) - \tan^{-1} \left( \frac{a}{u_j z} \right) \quad (j = 1, 2), & A_{ij} (i,j = 1-6)
\end{align*}
\]

\( u_1 \) and \( u_2 \) can be referred to Equation (9).
Figure 5. Flow chart for computing the lateral stress caused by the presented loading cases using the calculation charts.

Table II. Elastic properties and root type for the isotropic and cross-anisotropic backfills ($E = 50$ MPa, $v = 0.3$).

<table>
<thead>
<tr>
<th>Soil type</th>
<th>$E / E'$</th>
<th>$v / v'$</th>
<th>$G / G'$</th>
<th>Root type</th>
</tr>
</thead>
<tbody>
<tr>
<td>Soil 1. Isotropy</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>Equal</td>
</tr>
<tr>
<td>Soil 2. Cross-anisotropy</td>
<td>0.2</td>
<td>1.0</td>
<td>1.0</td>
<td>Distinct</td>
</tr>
<tr>
<td>Soil 3. Cross-anisotropy</td>
<td>2</td>
<td>1.0</td>
<td>1.0</td>
<td>Complex</td>
</tr>
</tbody>
</table>
Figure 6. Effect of \( z/b \). Variation of ratio of anisotropic to isotropic lateral stress for Soil 2: (a) horizontal uniform strip load; (b) horizontal upward linear-varying strip load; and (c) horizontal upward nonlinear-varying strip load.

linear-varying strip load (Figure 7(b)) and a vertical upward nonlinear-varying strip load (Figure 7(c)) approaches 0.4416, with increasing \( a/b \) from 0 to 10. The magnitude of the ratio induced by each vertical strip load is within 0.6, except for Figure 7(a) with \( z/b = 1 \) and 2, and Figures 7(b) and (c) with \( z/b = 2 \) case. Figures 6 and 7 indicate that the lateral stress in the cross-anisotropic backfill (Soil 2) is nearly less than that in the isotropic one (Soil 1), when both backfills result from horizontal and vertical strip loads.
Figures 8(a)–(c) and 9(a)–(c) illustrate the ratio of anisotropic to isotropic lateral stress \( \frac{\text{Soil 3}}{\text{Soil 1}} \) for the backfills subjected to the horizontal and vertical strip loads, respectively. These graphs show that as \( \frac{a}{b} \) rises from 0 to 10, the ratio reaches 0.9782 (Figures 8(a)–(c)) and 1.313 (Figures 9(a)–(c)), when the media are induced by a horizontal/vertical uniform strip load (Figure 8(a)/Figure 9(a)), a horizontal/vertical upward linear-varying strip load (Figure 8(b)/Figure 9(b)) and a horizontal/vertical upward nonlinear-varying strip load (Figure 8(c)/Figure 9(c)). Figures 8(a)–(c) for Soil 3 exhibit trends which are distinct from those in Figures 6(a)–(c).
for Soil 2. The same trends are true between Figures 9(a)–(c) and Figures 7(a)–(c), demonstrating that the lateral stress in the cross-anisotropic backfill (Soil 3), especially when induced by vertical strip loads, is almost greater than that in an isotropic backfill (Soil 1).

These analytical results reveal that the proposed solutions calculated the lateral stress in the cross-anisotropic backfills caused by horizontal/vertical uniform, linear-varying, and nonlinear-varying strip loads easily and accurately. Figures 6–9 also indicate that the effects of soil anisotropy (Soils 2 and 3), and the loading types (Cases C–E) on the lateral stress are explicit. Hence, in engineering practice, the lateral stress cannot be accurately estimated using traditional

Figure 8. Effect of z/b. Variation of ratio of anisotropic to isotropic lateral stress for Soil 3: (a) horizontal uniform strip load; (b) horizontal upward linear-varying strip load; and (c) horizontal upward nonlinear-varying strip load.
isotropic solutions, or by assuming that the surcharge strip load is uniformly distributed on a cross-anisotropic backfill.

CONCLUSIONS

Integrating the point load solution in a Cartesian co-ordinate system yields the analytical solutions for lateral stress caused by horizontal and vertical surcharge strip loads acting on a
cross-anisotropic backfill. The cross-anisotropic planes in this study are assumed to be parallel to the horizontal surface of the backfill. The surcharge loading types employed are horizontal/vertical point load, horizontal/vertical line load, horizontal/vertical uniform strip load, horizontal/vertical upward linear-varying strip load, horizontal/vertical upward nonlinear-varying strip load, horizontal/vertical downward linear-varying strip load and horizontal/vertical downward nonlinear-varying strip load. The presented stress solutions are significantly influenced by the type and degree of material anisotropy, and by the horizontal/vertical loading types. Furthermore, this work proposes two calculation charts to calculate the lateral stress induced by the presented horizontal/vertical strip loads if computers or calculators are unavailable. These charts are appropriate for cross-anisotropic backfills with two real distinct roots of the characteristic equation. The following conclusions can be drawn from the analytical results of a parametric study of two illustrative examples.

1. If the backfill material being Soil 2 \( (E/E' = 0.2, \nu/\nu' = G/G' = 1) \), with \( a/b \) increasing from 0 to 10, the ratio of anisotropic to isotropic lateral stress induced by horizontal and vertical strip loads, respectively, approaches 0.9506 and 0.4416.

2. Figures 6 and 7 show that the lateral stress caused by horizontal and vertical strip loads in Soil 2's backfill is nearly less than that in the isotropic backfill (Soil 1).

3. If the backfill material is Soil 3 \( (E/E' = 2, \nu/\nu' = G/G' = 1) \), then with \( a/b \) increasing from 0 to 10, the ratio of anisotropic to isotropic lateral stress induced by horizontal and vertical strip loads, respectively, approaches 0.9782 and 1.313.

4. Figures 8 and 9 demonstrate that the lateral stress, particularly when caused by vertical strip loads in the Soil 3's backfill, is greater than that of the isotropic backfill (Soil 1).

5. Figures 6–9 reveal that the effects of soil anisotropy (Soils 2 and 3) and the loading types (Cases C–E) on the lateral stress are explicit.

The solutions proposed herein can realistically simulate actual surcharge loading circumstances in many engineering practices. Moreover, the lateral stress resulting from any conceivable surcharge strip load can be analysed by superposing the presented loading cases. Regarding the expressions for the location of the centroid of total lateral stress and for the point of maximum unit lateral stress also can be obtained by integration. Similarly, this article can be extended to derive the lateral stress due to loading perpendicular to the retaining wall. Besides, the impact of the regular/irregular area loads of finite length [37, 38], or of a reinforced cross-anisotropic earth wall system subjected to the action of horizontal and vertical surcharge loads, can also be explored. The results of these investigations will be presented in forthcoming articles.

APPENDIX A: NOMENCLATURE

\( a \) load applied at a horizontal distance from the retaining wall
\( A_{ij} (i,j = 1–6) \) elastic moduli or elasticity constants
\( b \) width of the strip load
\( dx \) infinitesimal element along the \( x \)-axis
\( dy \) infinitesimal element along the \( y \)-axis
\( E \) Young’s modulus in the horizontal direction
\( E' \) Young’s modulus in the vertical direction
\( G' \) shear modulus in the vertical plane

\( H \)  
height of the retaining wall

\( i \)  
complex number \((= \sqrt{-1})\)

\( k, m_1, m_2, n \)  
coefficients

\( P \)  
a horizontal point load (force)

\( P^h \)  
a horizontal line load (force per unit length)

\( P^u \)  
the horizontal uniform strip load, and the maximum horizontal upward/downward linear-varying, and nonlinear-varying strip loads (force per unit area)

\( Q \)  
a vertical point load (force)

\( Q^v \)  
a vertical line load (force per unit length)

\( Q^u \)  
the vertical uniform strip load, and the maximum vertical upward/downward linear-varying, and nonlinear-varying strip loads (force per unit area)

\( s, t \)  
coefficients

\( u_1, u_2, u_3 \)  
roots of the characteristic equation

\( z \)  
vertical distance from point to load

\textbf{Greek letters}

\( v \)  
Poisson’s ratio for the effect of horizontal stress on complementary horizontal strain

\( v' \)  
Poisson’s ratio for the effect of vertical stress on horizontal strain

\( \sigma_{h_{\text{lin}}}^d \)  
lateral stress caused by a horizontal/vertical downward linear-varying strip load

\( \sigma_{h_{\text{non}}}^d \)  
lateral stress caused by a horizontal/vertical downward nonlinear-varying strip load

\( \sigma_h^l \)  
lateral stress caused by a horizontal/vertical line load

\( \sigma_h^p \)  
lateral stress caused by a horizontal/vertical point load

\( \sigma_h^{u_{\text{lin}}} \)  
lateral stress caused by a horizontal/vertical upward linear-varying strip load

\( \sigma_h^{u_{\text{non}}} \)  
lateral stress caused by a horizontal/vertical upward nonlinear-varying strip load

\textbf{ACKNOWLEDGEMENTS}

The author would like to thank Wan-Ling Chen and Ya-Ting Lin of National United University for their assistance.

\textbf{REFERENCES}


