Lateral stresses caused by uniform rectangular area loads on a cross-anisotropic backfill

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INTRODUCTION
Surcharge loads applied to the backfill of a retaining structure produce additional lateral stresses that, for working load levels, can be predicted using the theory of elasticity. These surcharge loads might arise from wheels, railway tracks, highway pavements or foundations of adjacent buildings, and could be modelled as point loads, line loads, strip loads or area loads. In a conventional elasticity calculation the backfill material is assumed to be a homogeneous, linearly elastic and isotropic continuum. Nevertheless, numerous studies have recognised that the elastic properties in the horizontal and vertical planes might be different. Hence the effect of anisotropic deformability on the lateral stresses induced by surcharge loading should be taken into consideration.

Poulos & Davis (1974) provided a large number of solutions for cross-anisotropic media, resulting from various types of surface load. This classical text also gave two extensive appendices (by Gerrard & Harrison) containing solutions for circular loaded areas. These solutions are useful, as the effect of a circular loaded region could be practically the same as that of a square one. A detailed study of surcharge-induced lateral stresses on rigid retaining structures with cross-anisotropic backfill was made by Wang (2005), who derived analytical solutions for various horizontal and vertical surcharge loads. These included a horizontal/vertical point load, a horizontal/vertical infinite line load, a horizontal/vertical uniform strip load, and various linear and non-linear varying strip loads. However, in the case of a backfill with an irregularly shaped loaded area the strip loading solution proposed by Wang (2005) might not be very suitable. For this reason, the present article describes analytical solutions for the lateral stresses caused by uniform horizontal and vertical rectangular area loads on a cross-anisotropic backfill. An arbitrary irregularly shaped area can be handled by approximating it with a suitable number of rectangles, and using superposition.

The backfill material is assumed to be homogeneous, linearly elastic and cross-anisotropic. The retaining wall is vertical, with horizontal backfill, and the planes of cross-anisotropy are parallel to the surface of the backfill. Two further simplifying assumptions are made: (a) the wall does not move; and (b) the wall is perfectly smooth (there is no shear stress between the wall and the soil). Under these conditions, the lateral stress induced on the wall can be calculated by considering an elastic half-space carrying two loads of equal magnitude (e.g. Fang, 1991). The imaginary load would cause equal and opposite normal displacements on a plane midway between it and the real surcharge load, thus enforcing the desired zero-horizontal-displacement boundary condition at the retaining wall. Consequently, the horizontal stress on the wall is twice that induced in an elastic half-space (Fang, 1991). However, in reality a large stress concentration might be developed around the lower corner of a retaining wall in contact with the backfill.

Additionally, the theory of elasticity utilised in this investigation does not consider the strength of the soil or the variation of its stiffness under different stress states. Also, the assumption of a perfectly smooth wall is restrictive, and limits the applicability of the elasticity method to practical applications. Nevertheless, a series of experiments conducted at the Iowa Engineering Experiment Station (Spangler, 1936, 1938a, 1938b; Spangler & Mickel, 1956; Spangler & Handy, 1982) and by Terzaghi (1954) confirmed the fact that doubling the horizontal stress in an elastic half-space could provide a good approximation to measured values of earth pressures on retaining walls (Fang, 1991).

In this study, lateral stress solutions for rectangular area loads applied to a cross-anisotropic backfill are obtained by integrating the point load solutions of Wang & Liao (1999). Fig. 1(a) depicts an applied horizontal/vertical point load, \( P/Q \), acting at a point with coordinates \( x = a, y = c, z = 0 \) relative to a cross-section through the retaining wall, whose height \( H \) is assumed to be large relative to \( a \) and \( c \). Once solutions for the point load case have been obtained, they are used to generate solutions for the lateral stress induced by a horizontal/vertical finite line load (Fig. 1(b)), and a horizontal/vertical uniform rectangular area load (Fig. 1(c)). Illustrative examples are then used to clarify the influence of the type and degree of material anisotropy, the loading distances from the retaining wall (\( a \) and \( c \)), the dimensions of the loaded area (\( l \) and \( w \)), and the loading direction (horizontal or vertical).

CASE A: LATERAL STRESS CAUSED BY A HORIZONTAL/VERTICAL POINT LOAD
Referring to Fig. 1(a), the exact solution for the horizontal stress \( \sigma_{xz} \) due to a horizontal point load \( P \) and a vertical point load \( Q \) on the surface of a cross-anisotropic half-space can be recast from Wang & Liao (1999) as

\[ \sigma_{xz} = \frac{P}{Q} \left[ \frac{a^2}{a^2 + c^2} \right] \]

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Fig. 1. Lateral stress caused by three types of horizontal/vertical surcharge loads on a cross-anisotropic backfill: (a) point load case; (b) finite line load case; (c) uniform rectangular area load case

\[ \sigma_{xx} = \frac{P}{2\pi R^2} \left( \frac{k(m_2u_1 - m_1u_2)}{m_1m_2(u_1 - u_2)} \right) \left[ A_{44}(u_1^2p_{s11} - u_2^2p_{s12}) - 2A_{46} \left( \frac{u_1}{m_1 + u_1} p_{s21} - \frac{u_2}{m_2 + u_2} p_{s22} \right) \right] + 2u_3p_{s23} \]

\[ + \frac{Q}{2\pi R} \left( \frac{k(m_2u_1 - m_1u_2)}{u_1 - u_2} \right) \left[ A_{44}(u_1p_{s11} - u_2p_{s12}) - 2A_{46} \left( \frac{1}{m_1 + u_1} (p_{s31} - p_{s41}) - \frac{1}{m_2 + u_2} (p_{s32} - p_{s42}) \right) \right] \]

For a perfectly smooth rigid retaining wall, the lateral stress \( \sigma_{kh}^s \) is assumed to be twice as large as that computed from the cross-anisotropic point load solution,

\[ \sigma_{kh}^s = 2 \frac{P}{2\pi R^2} \left( \frac{k(m_2u_1 - m_1u_2)}{m_1m_2(u_1 - u_2)} \right) \left[ A_{44}(u_1^2p_{s11} - u_2^2p_{s12}) - 2A_{46} \left( \frac{u_1}{m_1 + u_1} p_{s21} - \frac{u_2}{m_2 + u_2} p_{s22} \right) \right] + 2u_3p_{s23} \]

\[ + 2 \frac{Q}{2\pi R} \left( \frac{k(m_2u_1 - m_1u_2)}{u_1 - u_2} \right) \left[ A_{44}(u_1p_{s11} - u_2p_{s12}) - 2A_{46} \left( \frac{1}{m_1 + u_1} (p_{s31} - p_{s41}) - \frac{1}{m_2 + u_2} (p_{s32} - p_{s42}) \right) \right] \]

In this equation,

(a) \( A_{ij} (i, j = 1-6) \) are the elastic moduli or elasticity constants of the medium (Liao & Wang, 1998), and can be expressed in terms of the five independent elastic constants for a cross-anisotropic medium as

\[ A_{11} = \frac{E[1 - (E/E')v^2]}{(1 + v)(1 - 2v)(E/E')v^2} \]

\[ A_{13} = \frac{E'}{1 - v - (2E/E')v^2} \]

\[ A_{33} = \frac{E}{1 - v - (2E/E')v^2} \]

\[ A_{44} = G', \ A_{66} = \frac{E}{2(1 + v)} \]

where \( E \) is Young’s modulus in the horizontal direction, \( E' \) is Young’s modulus in the vertical direction, \( v \) is Poisson’s ratio for the effect of horizontal stress on complementary horizontal strain, \( v' \) is Poisson’s ratio for the effect of vertical stress on horizontal strain, and \( G' \) is the shear modulus in the vertical plane (Lee & Rowe, 1989).

(b) \( u_3 = \sqrt{A_{66}/A_{44}} \) and \( u_{1,2} \) are the roots of the characteristic equation

\[ u^4 - su^2 + t = 0 \]

where \( s = [A_{11}A_{33} - A_{13}(A_{13} + 2A_{44})]/A_{33}A_{44} \) and \( t = A_{11}/A_{33} \). As the strain energy is assumed to be positive definite in the medium, the values of elastic constants are restricted. Hence there are three categories of the characteristic roots \( u_{1,2} \), as follows.

Case 1

\[ u_{1,2} = \pm \frac{1}{2} \sqrt{s \pm \sqrt{(s^2 - 4t)}} \]

are two real distinct roots when \( s^2 - 4t > 0 \).

Case 2

\[ u_{1,2} = \pm \sqrt{s/2}, \pm \sqrt{-s/2} \]

are equal real roots when \( s^2 - 4t = 0 \) (i.e. complete isotropy).

Case 3

\[ u_1 = \frac{1}{2} \sqrt{s + 2\sqrt{t} - \frac{1}{2} \sqrt{s^2 - 4t}} = \gamma + i\delta, \]

\[ u_2 = \gamma - i\delta \]

are two complex conjugate roots when \( s^2 - 4t < 0 \).

(c) \( k = \frac{A_{11} + A_{44}}{A_{33}A_{44}(u_1^2 - u_2^2)} \)

\[ m_j = \frac{(A_{13} + A_{44})u_j}{A_{33}u_j^2} = \frac{A_{11} - A_{44}u_j^2}{A_{33}u_j^2} \]

\[ (A_{13} + A_{44})u_j \]

\[ (j = 1, 2); \]

\[ p_{s11} = \frac{x + a}{R_i^3} \]

\[ p_{s21} = \frac{x + a}{R_i^3} - \frac{3(x + a)}{R_i(R_i + z_i)^2} + \frac{(x + a)^3}{R_i(R_i + z_i)^3} \]

\[ p_{s13} = \frac{z_i}{R_i^3} \]

\[ p_{s23} = \frac{1}{R_i(R_i + z_i)^2} - \frac{(x + a)^3}{R_i(R_i + z_i)^3} \]
The analytical solution for the lateral stress, \( \sigma_{ky}^{(i)} \), can be derived by integrating the elementary stress functions of the point load solution \( (P_{i1}, P_{i2}) \) with respect to \( y \) between the limits 0 and \( c \), as follows.

\[
\begin{align*}
d_{41i} &= \int_0^c \left[ \frac{P_{i1}}{c} \left( \frac{x}{(x+a)^2 + c^2} \right) - \frac{P_{i2}}{c} \left( \frac{x}{(x+a)^2 + (c+l)^2} \right) \right] dy \\
d_{42i} &= -\left( x+a \right) \int_0^c \left[ \frac{c}{(x+a)^2 + c^2} \left( \frac{1}{\sqrt{(x+a)^2 + c^2}} \right)^2 - \frac{c}{(x+a)^2 + (c+l)^2} \left( \frac{1}{\sqrt{(x+a)^2 + (c+l)^2}} \right)^2 \right] dy \\
d_{43i} &= \int_0^c \left[ \frac{z_i}{(x+a)^2 + c^2} - \frac{c}{(x+a)^2 + (c+l)^2} \right] dy \\
d_{44i} &= \int_0^c \left[ \frac{z_i}{(x+a)^2 + c^2} \right] dy
\end{align*}
\]

The explicit solution for the lateral stress due to a finite line load can be regrouped in the same form as equation (2), except that \( P^*Q \) and the elementary stress functions \( p_{i1}, p_{i2} \) \((i = 1, 2, 3)\) are replaced, respectively, by \( P^*Q \) and the stress integral functions \( d_{41i} - d_{44i} \) \((i = 1, 2, 3)\), as follows.

\[
\begin{align*}
e_{31i} &= \tan^{-1} \left( \frac{\frac{n_{1i}n_{2i}}{\sqrt{n_{1i}^2 + n_{2i}^2} + 1}}{\sqrt{n_{1i}^2 + n_{2i}^2}} \right) \frac{n_{1i}|n_{2i}|}{\sqrt{n_{1i}^2 + n_{2i}^2} + 1} \\
e_{32i} &= \tan^{-1} \left( \frac{n_{1i}n_{2i}}{\sqrt{(n_{1i} + n_{2i})^2 + n_{3i}^2} + 1} \right) \\
e_{33i} &= \tan^{-1} \left( \frac{n_{1i}n_{2i}}{\sqrt{n_{1i}^2 + n_{2i}^2} + (n_{3i} + n_{4i})^2} + 1 \right) \\
e_{34i} &= \tan^{-1} \left( \frac{n_{1i}n_{2i}}{\sqrt{(n_{1i} + n_{2i})^2 + (n_{3i} + n_{4i})^2} + 1} \right)
\end{align*}
\]
Effect of $c/w$ caused by horizontal uniform rectangular load for isotropic/cross-anisotropic backfills:
- $c/w = 0$ for $E/E' = 0.2$, $v/v' = G/G' = 1$ (Soil 1)
- $c/w = 1$ for $E/E' = 0.2$, $v/v' = G/G' = 1$ (Soil 1)
- $c/w = 5$ for $E/E' = 0.2$, $v/v' = G/G' = 1$ (Soil 1)
- $c/w = 0$ for $E/E' = 1$, $v/v' = G/G' = 1$ (Soil 2)
- $c/w = 1$ for $E/E' = 1$, $v/v' = G/G' = 1$ (Soil 2)
- $c/w = 5$ for $E/E' = 1$, $v/v' = G/G' = 1$ (Soil 2)
- $c/w = 0$ for $E/E' = 2$, $v/v' = G/G' = 1$ (Soil 3)
- $c/w = 1$ for $E/E' = 2$, $v/v' = G/G' = 1$ (Soil 3)
- $c/w = 5$ for $E/E' = 2$, $v/v' = G/G' = 1$ (Soil 3)

Fig. 2. Effect of $c/w$ on induced lateral stress caused by: (a) horizontal rectangular area load; (b) vertical rectangular area load.
Fig. 3. Effect of $a/l$ on induced lateral stress caused by: (a) horizontal rectangular area load; (b) vertical rectangular area load
where $n_{3i} = a/z_i$, $n_{52} = l/z_i$, $n_{3i} = c/z_i$ and $n_{4i} = w/z_i$ ($i = 1, 2, 3$) are non-dimensional factors (functions of $a$, $c$, $l$, $w$ and $z$).

**ILLUSTRATIVE EXAMPLES**

This section presents a parametric study to confirm the proposed solutions and to demonstrate the effect of the type and degree of backfill anisotropy, the loading distances from the retaining wall, the dimensions of the loaded area, and the loading directions, on the lateral stress. Concerning typical ranges of cross-anisotropic elasticity parameters, Gazetas (1982) summarised experimental data regarding deformational cross-anisotropy of clays and sands, and concluded that the ratio $E/E'$ ranged from 0.6 to 4 for clays, and as was low as 0.2 for sands. Table 1 lists the elastic properties and root type of the three hypothetical backfill materials used in this parametric study. Soils 1 and 3 are cross-anisotropic, whereas Soil 2 is isotropic. The values adopted for $E$ and $v$ are 50 MPa and 0.3 respectively.

Based on the derived equations, a Mathematica program was written to calculate the induced lateral stress under a point load (equation (2)), a finite line load (equations (2), (6)–(9)), and a uniform rectangular area load (equations (2), (11)–(14)). The results discussed here are the lateral stresses caused by horizontal and vertical uniform rectangular area loads, for Soils 1–3.

Figures 2(a) and 2(b) depict the effect of the non-dimensional ratio $n_{3i}/n_{4i} = (c/w)/(w/z_i) = c/w = 0$, 1, 5 on the induced lateral stress for horizontal and vertical rectangular area loads respectively. The two figures show that the induced non-dimensional lateral stresses $\sigma_{3i}^u/P^u$ (caused by a horizontal load) and $\sigma_{4i}^u/Q^u$ (caused by a vertical load) increase with increasing $E/E'$ (Soil 1 → Soil 3), and decrease with increasing $c/w$ (from 0 → 5). It is also observed that the influences of soil anisotropy on $\sigma_{3i}^u/P^u$ and $\sigma_{4i}^u/Q^u$ become less apparent as $c/w$ increases. The figures demonstrate that the lateral stresses are intensely affected by the type and degree of soil anisotropy ($E/E'$), the value of $c/w$, and the loading direction (horizontal or vertical).

Similarly, Figs 3(a) and 3(b) show the effect of the other non-dimensional ratio $n_{3i}/n_{52} = (a/l)/(l/z_i) = a/l = 0$, 1, 5 on the induced lateral stress for horizontal and vertical rectangular area loads respectively. From Fig. 3(a) it can be seen that $\sigma_{3i}^u/P^u$ increases with decreasing $E/E'$ in the case of $a/l = 0$; however, when $a/l = 1$ and 5, the orders of $\sigma_{3i}^u/P^u$ are reversed. Fig. 3(b) indicates that $\sigma_{4i}^u/Q^u$ increases with increasing $E/E'$ for all $a/l$.

Figures 2 and 3 clearly reveal the influences of the type and degree of material anisotropy ($E/E'$), the loading distances from the retaining walls ($a$, $c$), the dimensions of the loaded area ($l$, $w$), and the loading direction (horizontal or vertical) on the lateral stress. They also show that when a uniform rectangular area load is applied to a cross-anisotropic backfill, the lateral stress cannot be accurately computed by using previous solutions for strip loading (Wang, 2005).

**CONCLUSIONS**

In this study, analytical solutions have been developed for the horizontal stress on a smooth, rigid retaining wall with cross-anisotropic elastic backfill, due to uniformly distributed horizontal and vertical loads applied over a rectangular area on the surface of the backfill. The planes of cross-anisotropy are assumed to be parallel to the horizontal surface of the backfill. It has been shown that the calculated lateral stresses are profoundly affected by the type and degree of soil anisotropy, the loading distances from the retaining wall, the dimensions of the loaded area, and the loading direction. In particular, illustrative examples have been given for two hypothetical cross-anisotropic sands (Soils 1 and 3), and an isotropic sand (Soil 2).

The lateral stress resulting from an irregularly shaped area load can be computed by dividing the loaded area into many rectangles; influences from these sub-areas are then superimposed. The present solutions could offer a valuable reference for the design of relatively rigid retaining structures under working surcharge loads.

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