Three-Dimensional Nonlinearly Varying Rectangular Loads on a Transversely Isotropic Half-Space

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Abstract: In engineering situations, loads applied to the four corners of a rectangle might have different values and might not be uniformly or linearly distributed. A configuration of linearly or nonlinearly varying loads with different contact pressures at each corner can be represented as a superposition of various loading types. The loading types include uniform, linearly varying in the x direction, linearly varying in the y direction, nonlinearly varying in the x direction, and nonlinearly varying in the y direction. This work newly presents the first and second loading solutions, and derives the others therefrom. These solutions are directly obtained by integrating the point load solutions in a transversely isotropic half-space. The presented solutions are concise and easy to use; they specify that the type and degree of material anisotropy, the dimensions of the loaded region, and the loading types decisively affect the displacements and stresses in a transversely isotropic half-space. The proposed solutions can simulate realistically the actual loading problem in many engineering situations.


CE Database subject headings: Displacement; Three-dimensional analysis; Stress; Half space; Load factors.

Introduction

Theoretical analyses of soil/rock behavior normally assume that the soil/rock is isotropic. However, many natural soils are deposited by geological sedimentation over a period, such as flocculated clays, or rock masses cut by discontinuities, which are not isotropic. Under such circumstances, in calculating the displacements and stresses induced by the applied loads, anisotropic deformability must be considered. This work addresses the elastic loading problem for a transversely isotropic half-space.

The solutions of displacements and stresses for various types of applied loads to isotropic/anisotropic half-spaces are important in designing foundations. A point load solution is well known to be the basis of complex loading problems for all constituted materials. However, external loads are more complex point loads in most engineering situations. Hence, closed-form solutions for displacements and stresses in a half-space when various loads are applied to different regions are required in engineering designs. Wang and Liao (1999, 2001) reviewed, in detail, regularly shaped loading solutions for the transversely isotropic half-space. Practicing engineers are commonly concerned with computing displacements and stresses in horizontal ground that bears a load uniformly distributed over a rectangular area (Giroud 1968). Nevertheless, in reality, applied loads are frequently not uniformly distributed (Hooper 1976; Bauer et al. 1979); also, the loads at the four corners of a rectangle could be different (Algin 2001), as depicted in Figs. 1 and 2. Hence, the loads may be more realistically as simulated as linearly varying, or nonlinearly varying and quadratically distributed. For an isotropic medium, Jarquio and Jarquio (1983) proposed a direct method of dimensioning a rectangular footing area under biaxial bending. They showed that maximum and minimum stresses were developed at the critical corners while the stresses at the other pair of diagonally opposite corners were equal. However, Vitone and Valsangkar (1986) pointed out that their solutions were erroneous, and presented an arithmetical solution in the case of a linear varying loading using the principle of superposition. The loading intensity at one of the corners was zero, at two other corners was equal, and at the final corner was double that the pair of corners with the nonzero load. That one corner has a zero contact pressure does not necessitate that two other corners have equal pressure, or that the equal pressure is in turn equals one-half of the maximum load intensity at the opposite corner (Algin 2001). Restated, corner load intensities might be different in various circumstances. Nevertheless, the aforementioned studies estimated vertical stress in an isotropic half-space. Closed-form solutions for displacements and stresses induced by uniform and linearly varying rectangular loads in the x direction for a transversely isotropic half-space have been presented by the writer elsewhere (Wang and Liao 1999). Elastic solutions for displacements and stresses subjected to a right-angled triangular loading have also been derived (Wang and Liao 2001). Although such triangular solutions can be extended to the rectangular solutions by superposition, the former cannot be used to solve the case of a nonlinearly varying loading since they are limited to uniform loads, and loads that vary linearly in the x and y directions. Recently, solutions for displacements (Wang and Liao 2002a) and stresses (Wang and Liao 2002b) in a transversely isotropic half-space caused by an upwardly and a downwardly linearly varying load, and a concave and a convex parabolic load on a rectangle, have been proposed. However, these solutions cannot be extended to calculate the displacements and stresses because of different loads at each corner.

Fortunately, the problems involving linearly varying loads,
shown in Fig. 1, can be solved by applying the principle of superposition to solutions to the cases of uniform loads \((Wang \text{ and Liao 1999})\) [Fig. 1(b)], linearly varying loads in the \(x\) direction \((Wang \text{ and Liao 1999})\) [Fig. 1(c)], and linearly varying loads in the \(y\) direction [Fig. 1(d)]. Therefore, closed-form solutions for displacements and stresses in a transversely isotropic half-space under rectangular loads that vary linearly rectangular loads in the \(y\) direction are required. Moreover, nonlinearly varying loads, as illustrated in Fig. 2, can also be expressed as superpositions of uniform loads \((Wang \text{ and Liao 1999})\) [Figs. 2(b) or 1(b)], loads that vary nonlinearly in the \(x\) direction [Fig. 2(c)], and loads that vary nonlinearly in the \(y\) direction [Fig. 2(d)]. However, to the best of the writer’s knowledge, no closed-form solutions for displacements and stresses in a transversely isotropic medium subjected to three-dimensional nonlinearly varying loads quadratically distributed and acting on a rectangle, have been proposed. The method of integration is recognized to be the most readily understood means of deriving solutions, and to be relatively easy to implement \((Gray 1943)\). Hence, closed-form solutions for displacements and stresses in a transversely isotropic half-space due to three-dimensional linearly and nonlinearly varying rectangular loads can be obtained by integrating the point load solutions in a Cartesian coordinate system \((Wang \text{ and Liao 1999})\). The obtained solutions are concise and easy to use; also, they reveal that the type and degree of material anisotropy, the dimensions of the loaded region, and the loading types influence the displacements and stresses in a transversely isotropic half-space. Two examples are given to illustrate the derived solutions and to elucidate the effect of rock anisotropy, the dimensions of the loaded area, and the type of loading, on the vertical surface displacement and vertical normal stress in the isotropic/transversely isotropic rocks, under a nonlinearly varying rectangular load in the \(x\) and \(y\) directions.

**Linearly Varying Loading Solutions**

In this work, the solutions for displacements and stresses in a transversely isotropic half-space subjected to linearly and nonlinearly varying rectangular loads are directly obtained by integrating the point load solutions in a Cartesian coordinate system \((Wang \text{ and Liao 1999})\). The closed-form solutions for displacements and stresses under three-dimensional point loads \((P_x, P_y, P_z)\) acting at \(z=h\) (from the surface) in the interior of a transversely isotropic half-space are rewritten in the Appendix. It means that in the following derivations, the three-dimensional linear/nonlinearly varying loads cannot only be applied on the surface but also in the interior of a transversely isotropic half-space. In the case of point load solutions, defining \(P_{d1}, P_{d0}\) in Eqs. (75)–(77) and \(P_{d1}, P_{d0}\) in Eqs. (78)–(83) as the elementary functions for displacements and stresses, respectively. Then, the
solutions for displacements and stresses in a transversely isotropic half-space due to linearly and nonlinearly varying rectangular loads can be directly integrated from the elementary functions of the point load solutions. The closed-form solutions for displacements and stresses induced by uniform and linearly varying loads over a rectangular region are presented first.

Fig. 1(a) shows a transversely isotropic half-space subjected to three-dimensional buried linearly varying loads on a rectangle ($q_{aj}^{lin}$, Fig. 1(b)), linearly varying loads in the $x$ direction ($q_{aj}^{lin}$, Fig. 1(c)), and linearly varying loads in the $y$ direction ($q_{bj}^{lin}$, Fig. 1(d)). The constant load intensities $q_{aj}^{lin}$, $q_{bj}^{lin}$, and $q_{dj}^{lin}$ ($j = x, y, z$) at the three vertices $(a, b, d)$ of a rectangle are expressed as

$$
\begin{align*}
\alpha &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} q_{aj}^{lin} \\ q_{bj}^{lin} \\ q_{dj}^{lin} \end{bmatrix} \\
\beta &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} q_{aj}^{lin} \\ q_{bj}^{lin} \\ q_{dj}^{lin} \end{bmatrix}
\end{align*}
$$

At the final corner, $c$ = assumed maximum pressure in Fig. 1(a), and can be represented as

$$
q_{dij}^{lin} = q_{aj}^{lin} + q_{bj}^{lin} + q_{dj}^{lin}
$$

The closed-form solutions for displacements and stresses at any point of Fig. 1(a) can be derived as superpositions of (1) uniform loads $q_{aj}^{lin}$ ($j = x, y, z$) [Fig. 1(b)], (2) linearly varying loads in the $x$ direction, $q_{aj}^{lin}$ ($j = x, y, z$) [Fig. 1(c)], and (3) linearly varying loads in the $y$ direction, $q_{bj}^{lin}$ ($j = x, y, z$) [Fig. 1(d)]. Namely, in order to obtain the complete solutions of displacements and stresses for the load configuration of Fig. 1(a), it is imperative to derive those solutions induced by above three mentioned loads [Figs. 1(b–d)].

By the same approaches of Wang and Liao (1999), the closed-form solutions for displacements and stresses induced by $q_{aj}^{lin}$, $q_{bj}^{lin}$, and $q_{dj}^{lin}$ can be derived by integrating the elementary functions $p_{dij}^{rds}$ [Eqs. (75)–(77)], and $p_{sij}^{rds}$ [Eqs. (78)–(83)]. The explicit solutions of displacements and stresses for those loading types also can be regrouped as the forms of Eqs. (75)–(83). For example, the closed-form solutions for applied uniform rectangular loads ($q_{dij}^{lin}$) are the same as Eqs. (75)–(83) except that the displacement elementary functions $p_{dij}^{rds}$ and stress elementary functions $p_{sij}^{rds}$ are replaced by the displacement integral functions $r_{dij}^{rds}$ and stress integral functions $r_{sij}^{rds}$. The same is true for linearly varying rectangular loads in the $x$ and $y$ directions. Hence, only the displacement and stress integral functions for uniform rectangular loads, linearly varying rectangular loads in the $x$ direction, and linearly varying rectangular loads in the $y$ direction, are presented.

1. Loading case of Fig. 1(b): $r_{dij}^{rds}$ and $r_{sij}^{rds}$ are the displacement and stress integral functions induced by uniform loads (Wang and Liao 1999), which are rewritten in terms of more simple forms as follows:

$$
r_{dij}^{rds} = -xD_1 + x^2 D_2 + zD_3 - D_4
$$

![Fig. 2. Three-dimensional buried nonlinearly varying loads on a rectangle: (a) Uniform and nonlinearly varying loads in the $x$ and $y$ direction; (b) uniform loads; (c) nonlinearly varying loads in the $x$ direction; and (d) nonlinearly varying loads in the $y$ direction](image)
where

\[ r_{d3i} = -yD_5 + y* D_6 + z_4(D_7 - D_8) \]  
\[ r_{d3i} = -z(D_9 - D_{10}) - (R_i - R_{x*y*y} + R_{x*y}) \]  
\[ r_{d4i} = -xD_3 + x* D_4 - yD_9 + y* D_{10} - z(D_1 - D_2) \]  
\[ r_{d5i} = -xD_{11} + x* D_{12} - z(D_3 - D_4) - yD_7 + y* D_6 \]  
\[ r_{d6i} = r_{d3i} + r_{d2i} \]  
\[ r_{s1i} = D_1 - D_2 \]  
\[ r_{s2i} = D_3 - D_6 \]  
\[ r_{s3i} = \tan^{-1}\frac{xy}{z_iR_i} - \tan^{-1}\frac{x* y}{z_iR_{x*y}} \]  
\[ - \tan^{-1}\frac{xy*}{z_iR_{x*y}} + \tan^{-1}\frac{x* y*}{z_iR_{x*y*y}} \]  
\[ r_{s4i} = D_{11} - D_{12} \]  
\[ r_{s5i} = -D_3 + D_4 \]  
\[ r_{s6i} = r_{s3i} - r_{s5i} \]  
\[ r_{s7i} = -y\left(\frac{1}{R_i + z_i} - \frac{1}{R_{x*y} + z_i}\right) \]  
\[ + y*\left(\frac{1}{R_{x*y} + z_i} - \frac{1}{R_{x*y*y} + z_i}\right) \]  
\[ r_{s8i} = -x\left(\frac{1}{R_i + z_i} - \frac{1}{R_{x*y} + z_i}\right) \]  
\[ + x*\left(\frac{1}{R_{x*y} + z_i} - \frac{1}{R_{x*y*y} + z_i}\right) \]  

where

\[ x* = x - l \]  
\[ y* = y - w \]  
\[ R_i = \sqrt{x^2 + y^2 + z_i^2} \]  
\[ R_{x*y} = \sqrt{x^2 + y^2 + z_i^2} \]  
\[ R_{x*y*y} = \sqrt{x^2 + y^2 + z_i^2} \]  
\[ D_1 = \ln\left|\frac{R_{x*y} + y*}{R_i + y}\right| \]  
\[ D_2 = \ln\left|\frac{R_{x*y*y} + y*}{R_{x*y} + y}\right| \]  
\[ D_3 = \tan^{-1}\frac{x^2 + yz_i(R_i + z_i)}{xy} - \tan^{-1}\frac{x^2 + yz_i(R_{x*y} + z_i)}{xy*} \]  
\[ D_4 = \tan^{-1}\frac{x^2 + yz_i(R_{x*y} + z_i)}{x*y} - \tan^{-1}\frac{x^2 + yz_i(R_{x*y*y} + z_i)}{x*y*} \]  
\[ D_5 = \ln\left|\frac{R_{x*y} + y}{R_i + y}\right| \]  
\[ D_6 = \ln\left|\frac{R_{x*y*y} + y}{R_{x*y} + y}\right| \]  
\[ D_7 = \tan^{-1}\frac{y^2 + z_i(R_i + z_i)}{xy} - \tan^{-1}\frac{y^2 + z_i(R_{x*y} + z_i)}{xy*} \]  
\[ D_8 = \tan^{-1}\frac{y^2 + z_i(R_{x*y} + z_i)}{xy*} - \tan^{-1}\frac{y^2 + z_i(R_{x*y*y} + z_i)}{x*y*} \]  
\[ D_9 = \ln\left|\frac{R_{x*y} + z_i}{R_i + z_i}\right| \]  
\[ D_{10} = \ln\left|\frac{R_{x*y*y} + z_i}{R_{x*y} + z_i}\right| \]  
\[ D_{11} = \ln\left|\frac{R_{x*y} + z_i}{R_i + z_i}\right| \]  
\[ D_{12} = \ln\left|\frac{R_{x*y} + z_i}{R_{x*y} + z_i}\right| (i = 1, 2, 3, a, b, c, d, e) \]  

2. Loading case of Fig. 1(c): \( l_{d3i}^{t} - l_{d4i}^{t} \) and \( l_{s3i}^{t} - l_{s4i}^{t} \) are the displacement and stress integral functions induced by linearly varying loads in the x direction (Wang and Liao 1999), also rewritten as

\[ l_{d3i}^{t} = x r_{d3i} + \frac{(x^2 + y^2)}{2} D_1 - \frac{(x^2 + y^2)}{2} + y(R_i - R_{x*y}) \]  
\[ - \frac{y}{2}(R_i - R_{x*y}) + y(R_i - R_{x*y}) \]  
\[ - z_i(yD_9 - y* D_{10}) \]  
\[ l_{d4i}^{t} = x r_{d4i} - y(R_i - R_{x*y}) + y* (R_{x*y} - R_{x*y*y}) \]  
\[ - z_i(yD_9 - y* D_{10}) \]  
\[ l_{s3i}^{t} = x r_{s3i} + \frac{x^2}{2}(R_i - R_{x*y}) \]  
\[ - \frac{x^2}{2}(R_i - R_{x*y}) + \frac{y^2}{2} D_5 \]  
\[ - \frac{(y^2 - z_i^2)}{2} D_7 - z_i(yD_7 - y* D_8) \]  
\[ l_{s4i}^{t} = x r_{s4i} - \frac{y}{2}(R_i - R_{x*y}) \]  
\[ - \frac{y}{2}(R_i - R_{x*y}) + \frac{x^2}{2} D_9 \]  
\[ - \frac{y^2}{2} D_7 + \frac{y^2}{2} D_8 \]  
\[ l_{d3i}^{t} = x r_{d3i} + \frac{x^2}{2} D_1 + \frac{x^2}{2} D_{12} + \frac{y^2}{2} D_9 + \frac{y^2}{2} D_{10} \]  
\[ l_{d4i}^{t} = x r_{d4i} + \frac{y^2}{2} D_7 - \frac{y^2}{2}(R_i - R_{x*y} - R_{x*y*y}) \]  
\[ l_{s3i}^{t} = l_{d3i}^{t} + l_{d4i}^{t} \]  
\[ l_{d4i}^{t} = l_{d3i}^{t} + l_{d4i}^{t} \]
\[ l_{s1i} = - r_{s2i} + x r_{s2i} + z i r_{s1i} \]  
(23) 
\[ l_{s2i} = - r_{s3i} + x r_{s3i} - z i r_{s2i} \]  
(24) 
\[ l_{s3i} = x r_{s3i} - z i r_{s1i} \]  
(25) 
\[ l_{s4i} = - r_{s4i} + x r_{s4i} - x D_{11} + x * D_{12} \]  
(26) 
\[ l_{s5i} = x r_{s5i} - z i r_{s1i} + x D_b + x * D_{10} \]  
(27) 
\[ l_{s6i} = l_{s3i} - l_{s5i} \]  
(28) 
\[ l_{s7i} = - r_{s2i} + x r_{s7i} + y (x / R_i + z_i - x / R_i + z_i) \]  
(29) 
\[ - y * (x / R_i + z_i - x / R_i + z_i) \]  
\[ l_{s8i} = x r_{s8i} - z_i r_{s4i} - y \left( \frac{1}{R_i + z_i} - \frac{1}{R_i + z_i} \right) \]  
(30) 
\[ l_{s9i} = y r_{s9i} - x (R_i - R_{s6i}) \]  
(31) 
\[ l_{s10i} = y r_{s10i} + x / 2 (R_i - R_{s7i}) \]  
(32) 
\[ l_{s11i} = x^2 / 2 D_1 - x^2 / 2 D_2 + y / 2 (R_i - R_{s8i}) \]  
(33) 
\[ l_{s12i} = y r_{s12i} + z_i r_{s10i} - z_i (x D_1 - x * D_2) \]  
(34) 
\[ l_{s13i} = y r_{s13i} + (R_i - R_{s7i} - R_{s6i}) \]  
(35) 
\[ l_{s14i} = y r_{s14i} + (R_i - R_{s7i} + R_{s6i}) \]  
(36)
concentrated force \( P_f \) (\( x,y,z \)) by \( q_{ij}^{\text{con}}(\zeta/l)^2 \, d\zeta \, d\eta \) [Fig. 2(c)], and \( q_{ij}^{\text{arc}}(\eta/w)^2 \, d\zeta \, d\eta \) [Fig. 2(d)], \( y \) by \( (y-\eta) \) and \( x \) by \( (x-\zeta) \) in Eqs. (75)–(83), the solutions of displacements and stresses for the elementary force acting in the half-space are obtained. Then, integrating the solutions induced by the elementary forces with \( \eta \) from 0 to \( w \), and \( \zeta \) from 0 to \( l \) can derive the complete solutions for the cases of Figs. 2(c and d). That is to say, only the displacement and stress integral functions for nonlinearly varying rectangular loads in the \( x \) and \( y \) directions, will be presented.

4. Loading case of Fig. 2(c): \( n_{s1i}^x-n_{s3i}^x \) and \( n_{s1i}^y-n_{s3i}^y \) are the displacement and stress integral functions induced by nonlinearly varying loads in the \( x \) direction as:

\[
 n_{s1i}^x = x^2 r_{d1i} - 2x l_{d1i} + \frac{z^2}{3} r_{s1i} + lw z_i - \frac{1}{3} (xy R_i - x^* y R_{x+1})
 - xy R_{y+1} + x^* y R_{x+y+1}) - \frac{x^3}{3} D_1 + \frac{x^3}{3} D_2
 - \frac{y^2 (x^2 - z^2)}{3} D_3 + \frac{y^2 (y^2 - z^2)}{3} D_6 + z_i (y^2 D_1 - y^2 D_6)
\]

\[ \text{(46)} \]

\[
 n_{s3i}^x = x^2 r_{d3i} - 2x l_{d3i} - lw z_i
 + \frac{1}{2} [x(y R_i - y^* R_{x+1}) - x^* (y R_{x+1} - y^* R_{x+y+1})]
 + \frac{y (y^2 - z^2)}{2} D_3 + \frac{y (y^2 - z^2)}{2} D_6
 \times D_6 - z_i (y^2 D_1 - y^2 D_6)
\]

\[ \text{(47)} \]

\[
 n_{s1i}^y = x^2 r_{d1i} - 2x l_{d1i} + z_i (y^2 D_0 - y^2 D_{10})
 - \frac{1}{2} [(x^2 - 2y^2 + z_i^2) R_i - (x^2 - 2y^2 + z_i^2) R_{x+1}]
 - (x^2 - 2y^2 + z_i^2) R_{x+y+1} + (x^2 - 2y^2 + z_i^2) R_{x+y+1})
\]

\[ \text{(48)} \]

\[
 n_{s3i}^y = x^2 r_{d3i} - 2x l_{d3i} + \frac{z^2}{3} r_{s3i} + \frac{w}{6} (x^2 - x^2)
 - \frac{2z_i}{3} [(y(R_i - R_{x+1}) - y^* (R_{x+y+1} - R_{x+y+1}))]
 - \frac{x^3}{3} D_3 + \frac{x^3}{3} D_4 + \frac{y^3}{3} D_6 - \frac{y^3}{3} D_{10}
\]

\[ \text{(49)} \]

5. Loading case of Fig. 2(d): \( n_{s1i}^x-n_{s3i}^x \) and \( n_{s1i}^y-n_{s3i}^y \) are the displacement and stress integral functions induced by nonlinearly varying loads in the \( y \) direction as:

\[
 n_{s1i}^y = x^2 r_{d1i} - 2x l_{d1i} + \frac{w (y^2 + z_i^2)}{3} + \frac{z_i}{6} (x(R_i - R_{x+1}) - x^* (R_{x+1} - R_{x+y+1}))
 - \frac{z_i}{6} (3y^2 + z_i^2) D_5 + (3y^2 + z_i^2) D_6) + \frac{y^3}{3} D_7
 - \frac{y^3}{3} D_8 - \frac{x^3}{3} D_11 + \frac{x^3}{3} D_{12}
\]

\[ \text{(50)} \]

\[
 n_{s3i}^y = n_{s1i}^y + n_{s2i}^y
\]

\[ \text{(51)} \]

\[
 n_{s2i}^y = (x^2 - z_i^2) r_{s2i} - 2x l_{s2i} + y(R_i - R_{x+1})
 - y^* (R_{x+y+1} - R_{x+y+1})
\]

\[ \text{(52)} \]
\[ n_{d_1}^y = y^2 r_{d_1} - 2y l_{d_1} + \frac{z_3}{3} r_{s_3} + lwz_4 \]
\[ - \frac{1}{3} \left[ x(y R_i - y^* R_{s_4}) - x^* (y R_{s_4} - y^* R_{s_4}) \right] \]
\[ - \frac{1}{3} \left[ x(x^2 - 3 z_1^2) D_1 + x^* (x^2 - 3 z_1^2) \right] \]
\[ \times D_2 + z(z_1^2 D_3 - x^2 D_4) - \frac{y^3}{3} D_4 + \frac{y^3}{3} D_6 \] (61)

\[ n_{d_1}^y = y^2 r_{d_1} - 2y l_{d_1} + z(z_1^2 D_{11} - x^2 D_{12}) \]
\[ + \frac{1}{3} \left[ (2x^2 - y^2 - z_1^2) R_i - (2x^2 - y^2 - z_1^2) R_{s_4} \right] \]
\[ - (2x^2 - y^2 - z_1^2) R_{s_4} + (2x^2 - y^2 - z_1^2) R_{s_4} \] (62)

\[ n_{d_1^y} = y^2 r_{d_1^y} - 2y l_{d_1^y} + \frac{z_3}{6} r_{s_3^y} + \frac{w}{3} (x^2 - x^2) \]
\[ + \frac{z_1}{6} \left[ y(R_i - R_{s_4}) - y^* (R_{s_4} - R_{s_4}) \right] \]
\[ + \frac{z_1}{3} \left[ (x^2 D_4 - x^2 D_2) + \frac{x^3}{3} D_3 \right] \]
\[ - \frac{x^3}{3} D_4 + \frac{y^3}{3} D_6 + \frac{y^3}{3} D_{10} \] (63)

\[ n_{d_1^y} = y^2 r_{d_1^y} - 2y l_{d_1^y} + \frac{z_3}{3} r_{s_3^y} + \frac{lw(y + y^*)}{6} \]
\[ - \frac{2z_1}{3} \left[ x(R_i - R_{s_4}) - x^* (R_{s_4} - R_{s_4}) \right] \]
\[ - \frac{y^3}{3} D_4 + \frac{y^3}{3} D_6 + \frac{x^3}{3} D_{11} - \frac{x^3}{3} D_{12} \] (64)

\[ n_{d_1^y} = n_{d_1}^y + n_{d_1}^y \] (65)

\[ n_{s_4}^y = \left( y^2 - \frac{z_1^2}{2} \right) r_{s_4} - 2y l_{s_4} + \frac{y}{2} (R_i - R_{s_4}) \]
\[ + \frac{y^*}{2} \left( R_{s_4} - R_{s_4} \right) - \frac{x^2}{2} D_1 + \frac{x^2}{2} D_2 \] (66)

\[ n_{s_4^y} = \left( y^2 - \frac{z_1^2}{2} \right) r_{s_4^y} - 2y l_{s_4^y} + x(R_i - R_{s_4}) \]
\[ - x^* \left( R_{s_4} - R_{s_4} \right) \] (67)

\[ n_{s_3^y} = \left( y^2 - \frac{z_1^2}{2} \right) r_{s_3^y} - 2y l_{s_3^y} + z(xD_1 - x^* D_2) \] (68)

\[ n_{s_4}^y = y^2 r_{s_4} - 2y l_{s_4} + z(R_i - R_{s_4} - R_{s_4} + R_{s_4}) \]
\[ - x^2 D_1 + x^2 D_2 \] (69)

\[ n_{s_3^y} = y^2 r_{s_3^y} - 2y l_{s_3^y} + lw + z(xD_1 - x^* D_2) \]
\[ + x^2 D_1 - x^2 D_2 \] (70)

\[ n_{s_3^y} = n_{s_3}^y - n_{s_3}^y \] (71)

Utilizing Eqs. (75)–(83), (3)–(44), and (46)–(73), and the following expression, one can also estimate the displacements and stresses at any point of Fig. 2(a)

\[ \left[ \begin{array}{l} U \\ \sigma \end{array} \right]_{\text{non}} = q_{ij}^{\text{lin}} \left[ \begin{array}{ll} U(p_{d_1} - p_{d_4}) & \text{are replaced by } r_{d_1} - r_{d_4} \\ \sigma(p_{d_1} - p_{d_4}) & \text{are replaced by } r_{d_1} - r_{d_4} \end{array} \right] q_{ij}^{\text{non}} \]
\[ + q_{ij}^{\text{non}} \frac{q_{i}^{\text{lin}}}{F} \left[ \begin{array}{ll} U(p_{d_1} - p_{d_4}) & \text{are replaced by } n_{d_1}^y - n_{d_4}^y \\ \sigma(p_{d_1} - p_{d_4}) & \text{are replaced by } n_{d_1}^y - n_{d_4}^y \end{array} \right] q_{ij}^{\text{non}} \]
\[ + q_{ij}^{\text{non}} \frac{q_{i}^{\text{lin}}}{w^2} \left[ \begin{array}{ll} U(p_{d_1} - p_{d_4}) & \text{are replaced by } n_{d_1}^y - n_{d_4}^y \\ \sigma(p_{d_1} - p_{d_4}) & \text{are replaced by } n_{d_1}^y - n_{d_4}^y \end{array} \right] q_{ij}^{\text{non}} \] (74)

where the superscripts non, lin, and q represent the displacement and stress components induced by nonlinearly varying loads with different intensities at each corner [Fig. 2(a)], uniform loads [Figs. 2(b) and 1(b)], nonlinearly varying loads in the x direction [Fig. 2(c)], and nonlinearly varying loads in the y direction [Fig. 2(d)].

A flow chart that illustrates the derived solutions for computing the displacements and stresses due to three-dimensional (1) uniform loads, (2) linearly varying loads in the x direction, (3) linearly varying loads in the y direction, (4) nonlinearly varying loads in the x direction, and (5) nonlinearly varying loads in the y direction, all acting on a rectangular area of a transversely isotropic half-space is presented in Fig. 3.

**Illustrative Examples**

Two illustrative examples as depicted in Figs. 4 and 5 are conducted to verify the solutions derived and elucidate the effect of the type and degree of material anisotropy, the dimensions of the loaded area, and the types of loading on the displacement and stress. The degree of material anisotropy including the ratios E/E', v/v', G/G', and the loading types including a vertical nonlinearly varying rectangular load in the x and y directions.
acting on the horizontal surface \((h=0)\) of the half-space, are investigated. Several types of isotropic and transversely isotropic rocks are considered as foundation materials. Their elastic properties are listed in Table 1 with \(E/E_8\) and \(G/G_8\) ranging between 1 and 3, and \(n/n_8\) varying between 0.75–1.5 (Gerrard 1975; Amadei et al. 1987). Also, the root type in Table 1 can be distinguished by Eqs. (86) and (87) (Wang and Liao 1998) in the Appendix. The values adopted of \(E\) and \(n\) are 50 GPa and 0.25, respectively.

Based on Eqs. (75)–(83), (3)–(44), and (46)–(74) for nonlinearly varying loading types, a FORTRAN program was written to calculate the displacements and stresses. This program can compute all the components of displacement and stress at any point in the half-space. However, the vertical displacement and stress are usually the most interesting quantities in foundation analysis. Hence, in this study, only the vertical surface displacement and vertical normal stress at/below the right corner (point C) of the loaded area were presented. Figs. 4 and 5 show the results for the half-space subjected to nonlinearly varying rectangular loads in the \(x\) and \(y\) direction, respectively. The normalized vertical surface displacement \(\left[ u_z^{\text{non}} / (L^2 \sigma_{zz}^{\text{non}}) \right] \) at point C induced by a nonlinearly varying load in the \(x\) direction, and a nonlinearly varying load in the \(y\) direction resting on the free surface versus the nondimensional ratio of the loaded side \((w/l)\), are presented in Figs. 4(a) and 5(a), respectively. The others in Figs. 4 and 5 show that the induced vertical stress at point C with depth \(z\) from the surface for different rock types, dimensions of the loaded area, and two different loading types as mentioned above, acting on the surface. The relation of two nondimensional factors, \(l/l_z\) versus \(\sigma_{zz}^{\text{non}} / \sigma_{zz}^{\text{non}}\) and \(l/l_z\) versus \(\sigma_{zz}^{\text{non}} / \sigma_{zz}^{\text{non}}\) is reported in Figs. 4(b–d) and Figs. 5(b–d), respectively. The other nondimensional factor \(w/z\) is adopted for investigating the influence of the dimensions of the loaded region on the vertical stress. The loads can be assumed as a strip load when the ratio of \(w/z\) approaching to infinity (\(\infty\); Feda 1978). Based on the results reported in Figs. 4 and 5, the effect of the type and degree of rock anisotropy, the dimensions of the loaded area, and the loading types on the displacement and stress is elucidated below.

Figs. 4(a) and 5(a) indicate that for a given shape, the vertical displacement on the surface increases with increasing \(E/E'\) (∧/\(\gamma' = G/G' = 1\), Rocks 1, 2, and 3), and \(E/E'\) (∧/\(\gamma' = G/G' = 1\), Rocks 1, 4, and 5), and \(G/G'\) (∧/\(\gamma' = G/G' = 1\), Rocks 1, 6 and 7). The increases of the ratio of \(E/E'\) and \(G/G'\) do have a great influence on the vertical displacement. It reflects that the vertical surface displacement increases with the increase of deformability.
in the direction parallel to the applied load. However, the variation of $v/v'$ has little effect on the vertical surface displacement. It also can be found that at point C, if the load intensity in the $y$ direction is not sufficiently large, the effect on the induced vertical surface displacement [Fig. 5(a)] is less than that of the nonlinearly varying load in the $x$ direction [Fig. 4(a)]. Figs. 4(b–d) and 5(b–d) plot the vertical stress induced by the nonlinearly varying rectangular load in the $x$ and $y$ direction for Rocks 1, 2, 3, Rocks 1, 4, 5, and Rocks 1, 6, 7 with variable nondimensional factors ($m, n$), respectively. From Figs. 4(b–d), the vertical stress ($\sigma_{zz}^{\text{non}}$) induced by nonlinearly varying rectangular load in the $x$ direction ($q_{dc}^{\text{non}}$) might be transferred by tension with the increase of $m$ for $n=0.1, 0.5, \text{ and } 1.0$; however, for the plane strain case ($n=\infty$), the medium will produce tensile stress within the whole loaded area of $m$. Therefore, the results of Figs. 4(b–d) indicate that the compressive or tensile vertical stress depends substantially on the type and degree of rock anisotropy, and the dimensions of the loaded area. It is also interesting to find the vertical stress induced by a nonlinearly varying rectangular load in the $y$ direction [Figs. 5(b–d)] differs from that induced by a nonlinearly varying rectangular load in the $x$ direction [Figs. 4(b–d)]. From Figs. 5(b–d), the magnitude of nondimensional vertical stress de-

**Fig. 4.** Effect of rock anisotropy on vertical surface displacement and vertical stress induced by a vertical nonlinearly varying rectangular load in the $x$ direction $q_{dc}^{\text{non}}$: (a) Vertical surface displacement for all rocks; (b) vertical stress for Rocks 1, 2, 3; (c) vertical stress for Rocks 1, 4, 5; and (d) vertical stress for Rocks 1, 6, 7.
creases with increasing $E/E'$ (Rocks 1, 2, 3), increases with increasing $G/G'$ (Rocks 1, 6, 7), but is little affected by the ratio of $v/v'$ (Rocks 1, 4, 5) for $n=0.1$, and 0.5. Nevertheless, the calculated result at point C will switch when $n=1.0$, and diminishingly zero for all rocks when $n=\infty$.

The above examples are presented to elucidate the solutions and clarify how the type and degree of rock anisotropy, the dimensions of the loaded area, and the loading types will affect the displacement and stress in the medium. Based on this study, the results show that the induced displacement and stress in isotropic/transversely isotropic rocks subjected to nonlinearly varying rectangular load in the $x$ and $y$ directions are easily calculated by the

![Fig. 5. Effect of rock anisotropy on vertical surface displacement and vertical stress induced by a vertical nonlinearly varying rectangular load in the $y$ direction $q_{bc\nu}$: (a) Vertical surface displacement for all rocks; (b) vertical stress for Rocks 1, 2, 3; (c) vertical stress for Rocks 1, 4, 5; and (d) vertical stress for Rocks 1, 6, 7.](image)

![Table 1. Elastic Properties and Root Types for Different Rocks](image)
Discussions and Conclusions

Based on the integration of elementary functions for a point load, the closed-form solutions of displacements and stresses in a transversely isotropic half-space under three-dimensional linearly varying and nonlinearly varying loads with different intensities at each corner of a rectangle, can be derived. The type and degree of material anisotropy (\(E/E', \nu/\nu', G/G'\)), the dimensions of the loaded area (\(l, w\)), and the type of loading (linearly or nonlinearly varying loads) are shown most strongly to influence the presented solutions. The closed-form solutions are the same as a few isotropic solutions if the medium is isotropic (Algin 2001). In particular, a parametric study of two illustrative examples was performed, yielding the following interesting conclusions.

1. The ratios \(E/E' (\nu/\nu' = G/G' = 1)\) and \(G/G' (E/E' = \nu/\nu' = 1)\) strongly influence the vertical surface displacement of transversely isotropic rocks subjected to a load that varies nonlinearly in the \(x\) and \(y\) directions; however, the effect of the ratio of \(\nu/\nu' (E/E' = G/G' = 1)\) is relatively small.

2. If the load intensity in the \(y\) direction is not sufficiently large, the effect on the induced vertical surface displacement is less than that of the nonlinearly varying load in the \(x\) direction.

3. The compressive or tensile vertical stress induced by a nonlinearly varying load in the \(x\) direction depends substantially on the type and degree of rock anisotropy, and the dimensions of the loaded area.

4. The vertical stress induced by a non-linearly varying rectangular load in the \(y\) direction differs from that induced by a nonlinearly varying rectangular load in the \(x\) direction.

The derived closed-form solutions are concise. Therefore, the estimation of displacements and stresses due to three-dimensional nonlinearly varying loads with different contact pressures at each corner of a rectangular area in an isotropic/transversely isotropic half-space is fast and accurate. The presented solutions can simulate realistically the actual loading problem in many areas of engineering practice.

Appendix

\[
\begin{align*}
\sigma^{\mu}_{xx} &= \frac{P_s}{4\pi m_1} \left[ k \frac{p_{d11}}{m_1} - k \frac{p_{d12}}{m_2} - T_1 p_{d1a} + T_2 p_{d1b} + T_3 p_{d1c} - T_4 p_{d1d} + \frac{1}{u_s A_{44}} \left( p_{d23} + p_{d24} \right) \right] \\
&\quad + \frac{P_s}{4\pi} \left[ - k \frac{p_{d31}}{m_1} + k \frac{p_{d32}}{m_2} - T_1 p_{d3a} - T_2 p_{d3b} - T_3 p_{d3c} + T_4 p_{d3d} + \frac{1}{u_s A_{44}} \left( p_{d33} + p_{d34} \right) \right] \\
&\quad - \frac{P_s}{4\pi} \left[ k \left( p_{d41} - p_{d42} \right) + m_1 \left( T_1 p_{d4a} - T_3 p_{d4c} \right) - m_2 \left( T_2 p_{d4b} + T_4 p_{d4d} \right) \right] \\
&\quad - \frac{P_s}{4\pi} \left[ k \left( p_{d41} - p_{d42} \right) + m_1 \left( T_1 p_{d4a} - T_3 p_{d4c} \right) - m_2 \left( T_2 p_{d4b} + T_4 p_{d4d} \right) \right] \\
&\quad + \frac{P_s}{4\pi} \left[ - k \left( p_{d51} - p_{d52} \right) + m_1 \left( T_1 p_{d5a} - T_3 p_{d5c} \right) - m_2 \left( T_2 p_{d5b} + T_4 p_{d5d} \right) \right] \\
&\quad - \frac{P_s}{4\pi} \left[ m_1 \left( k p_{d61} + T_1 m_1 p_{d6a} - T_2 m_2 p_{d6b} \right) - m_2 \left( k p_{d62} + T_3 m_1 p_{d6c} - T_4 m_2 p_{d6d} \right) \right] \\
&\quad \quad + P_s \left[ - 2A_{66} \left( \frac{k}{m_1} p_{s11} - \frac{k}{m_2} p_{s12} - T_1 p_{s1a} + T_2 p_{s1b} + T_3 p_{s1c} - T_4 p_{s1d} \right) + 2u_s (p_{s33} + p_{s77}) \right] \\
&\quad + P_s \left[ 2A_{66} \left( \frac{k}{m_1} p_{s11} - \frac{k}{m_2} p_{s12} - T_1 p_{s1a} + T_2 p_{s1b} + T_3 p_{s1c} - T_4 p_{s1d} \right) - 2u_s (p_{s33} + p_{s77}) \right] \\
&\quad + 2A_{66} \left( \frac{k}{m_1} p_{s11} - \frac{k}{m_2} p_{s12} - T_1 p_{s1a} + T_2 p_{s1b} + T_3 p_{s1c} - T_4 p_{s1d} \right) - 2u_s (p_{s33} + p_{s77}) \right],
\end{align*}
\]
\begin{align}
\sigma_{ij}^p &= \frac{P}{4\pi} \left( A_{11} - u_1 m_1 A_{13} - 2A_{66} \left( k p_{s1} + T_1 m_1 p_{s2} + T_2 m_2 p_{s5a} - (A_{11} - u_2 m_2 A_{13} - 2A_{66}) (k p_{s2} + T_1 m_1 p_{s6} + T_2 m_2 p_{s8}) \right) \right) \\
&+ 2A_{66} (k(p_{s1} - p_{s2}) + m_1(T_1 p_{s5a} - T_2 p_{s5c}) - m_2(T_2 p_{s5b} - T_4 p_{s5d})) \\
&+ P \left( A_{11} - u_1 m_1 A_{13} - 2A_{66} \left( k m_1 p_{s1} - T_1 p_{s1a} + T_2 p_{s1b} \right) \right) - (A_{11} - u_2 m_2 A_{13} - 2A_{66}) \left( k m_2 p_{s1} - T_3 p_{s1c} + T_4 p_{s1d} \right) \\
&+ 2A_{66} \left( k m_1 p_{s1} - T_1 p_{s1a} + T_2 p_{s1b} \right) - \frac{P}{4\pi} \left( A_{11} - u_1 m_1 A_{13} \right) \left( k m_1 p_{s1} - T_1 p_{s1a} + T_2 p_{s1b} \right) - (A_{11} - u_2 m_2 A_{13}) \left( k m_2 p_{s1} - T_3 p_{s1c} + T_4 p_{s1d} \right) \\
&+ 2A_{66} \left( k m_1 p_{s1} - T_1 p_{s1a} + T_2 p_{s1b} \right) - 2u_3 \left( p_{s1a} + p_{s1b} \right) \\
&+ P \left( A_{11} - u_1 m_1 A_{13} \right) \left( k m_1 p_{s1} - T_1 p_{s1a} + T_2 p_{s1b} \right) - (A_{11} - u_2 m_2 A_{13}) \left( k m_2 p_{s1} - T_3 p_{s1c} + T_4 p_{s1d} \right) \\
&+ 2A_{66} \left( k m_1 p_{s1} - T_1 p_{s1a} + T_2 p_{s1b} \right) - 2u_3 \left( p_{s1a} + p_{s1b} \right) \\
&+ P \left( A_{11} - u_1 m_1 A_{13} \right) \left( k m_1 p_{s1} - T_1 p_{s1a} + T_2 p_{s1b} \right) - (A_{11} - u_2 m_2 A_{13}) \left( k m_2 p_{s1} - T_3 p_{s1c} + T_4 p_{s1d} \right) \\
&+ 2A_{66} \left( k m_1 p_{s1} - T_1 p_{s1a} + T_2 p_{s1b} \right) - 2u_3 \left( p_{s1a} + p_{s1b} \right) \\
&+ \frac{P}{4\pi} \left( A_{11} - u_1 m_1 A_{13} \right) \left( k m_1 p_{s1} - T_1 p_{s1a} + T_2 p_{s1b} \right) - (A_{11} - u_2 m_2 A_{13}) \left( k m_2 p_{s1} - T_3 p_{s1c} + T_4 p_{s1d} \right) \\
&+ 2A_{66} \left( k m_1 p_{s1} - T_1 p_{s1a} + T_2 p_{s1b} \right) - 2u_3 \left( p_{s1a} + p_{s1b} \right)
\end{align}

where

- The generalized Hooke's law for the transversely isotropic medium in a Cartesian coordinate system can express the following matrix form, in which the z axis be the rotation axis of elastic symmetry, x and y axes in the plane of transversely isotropy.
where $A_j$ ($i,j=1−6$) are the elastic moduli or elasticity constants of the medium. For a transversely isotropic material, the five engineering elastic constants, $E$, $E'$, $ν$, $ν'$, and $G'$ are defined as:

1. $E$ and $E'$ are Young’s moduli in the plane of transverse isotropy and in a direction normal to it, respectively.
2. $ν$ and $ν'$ are Poisson’s ratios characterizing the lateral strain response in the plane of transverse isotropy to a stress acting parallel or normal to it, respectively.
3. $G'$ is the shear modulus in planes normal to the plane of transverse isotropy. Hence, $A_{ij} (i,j=1−6)$ can be expressed in terms of these elastic constants as

$$A_{11} = \frac{E(1-\nu')}{(1+\nu)(1-2\nu-\nu'^2)}, \quad A_{13} = \frac{E\nu'}{1-\nu-\frac{2E}{E'}\nu'^2},$$

$$A_{33} = \frac{E'(1-\nu)}{1-\nu-\frac{2E}{E'}\nu'^2}, \quad A_{44} = G', \quad A_{66} = \frac{E}{2(1+\nu)}.$$

- $u_3 = \sqrt{A_{66}/A_{44}}, u_1$ and $u_2$ are the roots of the following characteristic equation:

$$u^4 - su^2 + q = 0 \quad (86)$$

where $s = (A_{11}A_{33} - A_{13}A_{13} + 2A_{44})/A_{13}A_{33}, q = A_{11}A_{33}$. There are three categories of the characteristic roots, $u_1$ and $u_2$ as follows:

Case 1. When $s^2 - 4q > 0$, $u_{1,2} = \pm \sqrt{s \pm \sqrt{s^2 - 4q}}/2$ are two real distinct roots.

Case 2. When $s^2 - 4q = 0$, $u_{1,2} = \pm \sqrt{s}/2, \pm \sqrt{s}/2$ are real double roots (i.e., complete isotropy); and

Case 3. When $s^2 - 4q < 0$, $u_1 = \sqrt{s+i\sqrt{s^2 - 4q}}, u_2 = -i\sqrt{s+i\sqrt{s^2 - 4q}}$ are two conjugate complex roots [where $\gamma$ cannot be equal to zero (Wang and Liao 2002a)].

Using engineering elastic constants, the following criterion (Wang and Liao 1998) also can distinguish the root type of Eq. (86).

$$\left(\frac{G}{G'}\right)^2(1+\nu) - \left(\frac{E}{E'}\right)^2\left[1 - \nu + \frac{E}{G'}\nu'\right] - 2\left(\frac{E}{E'}\nu'\right)^2 \geq 0, \quad \text{for Case 1}$$

$$= 0, \quad \text{for Case 2}$$

$$< 0, \quad \text{for Case 3} \quad (87)$$

- $m_j = \frac{(A_{11} + A_{44})u_{j}}{A_{33}u_j - A_{44}} = \frac{A_{11} - A_{44}u_j^2}{(A_{11} + A_{44})u_j} (j = 1, 2)$

$$k = \frac{(A_{11} + A_{44})}{A_{33}A_{44}(u_1 - u_2)}$$

$$T_1 = \frac{k}{m_1}u_1 + u_2, \quad T_2 = \frac{k}{m_2}u_2 - u_1, \quad T_3 = \frac{k}{m_2}(u_1 + m_1), \quad T_4 = \frac{k}{m_2}(u_1 + u_2)$$

$$P_{dii} = \frac{x^2}{R_i + z_i}, \quad P_{dii} = \frac{y^2}{R_i + z_i}$$

References


