Vertical stress distributions around batter piles driven in cross-anisotropic media

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SUMMARY

This work presents analytical solutions to compute the vertical stresses for a cross-anisotropic half-space due to various loading types by batter piles. The loading types are an embedded point load for an end-bearing pile, uniform skin friction, and linear variation of skin friction for a friction pile. The cross-anisotropic planes are parallel to the horizontal ground surface. The proposed solutions can be obtained by utilizing Wang and Liao’s solutions for a horizontal and vertical point load acting in the interior of a cross-anisotropic medium. The derived cross-anisotropic solutions using a limiting approach are in perfect agreement with the isotropic solutions of Ramiah and Chickanagappa with the consideration of pile inclination. Additionally, the present solutions are identical to the cross-anisotropic solutions by Wang for the batter angle equals to 0. The influential factors in yielded solutions include the type and degree of geomaterial anisotropy, pile inclination, and distinct loading types. An example is illustrated to clarify the effect of aforementioned factors on the vertical stresses. The parametric results reveal that the stresses considering the geomaterial anisotropy and pile batter differ from those of previous isotropic and cross-anisotropic solutions. Hence, it is imperative to take the pile inclination into account when piles are required to transmit both the axial and lateral loads in the cross-anisotropic media. Copyright © 2008 John Wiley & Sons, Ltd.

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INTRODUCTION

Batter piles (or referred to as inclined piles or raked piles) that are driven at a slope with respect to the vertical, could generally be chosen to support the offshore structures, bridges, and towers [1–7]. When significant lateral forces are to be resisted by a pile group, it has been a common practice to use the batter piles [8, 9]. However, lateral forces might be imposed by earth pressure, by the wind, by earthquakes, by braking and acceleration loads, by the impact of berthing ships, by the pull from mooring ropes, or by the pressure of currents, waves, and floating ice [2]. Typically, plumb piles (or referred to as vertical piles) have very low resistances to lateral forces, and consequently in this article, the vertical stress distributions resulting from batter piles driven into the cross-anisotropic geomaterials are investigated.

Practically, plumb/batter piles transmit axial/axial–lateral loads to nearby soil mainly by the mechanism of end bearing or skin friction. An end-bearing pile, which relies on the concentrated soil resistance at the tip of the pile, and at the tip’s resistance, could be modeled as a point load [10]. The load distributions around the pile shaft are due to the shear forces acting along the interface of the pile and soil and are called the skin friction piles [11]. These frictional loads could be simulated as uniformly or linearly varying distributed with depth from the ground surface to the pile length [11, 12]. According to the aforementioned loading cases, Wang [13] proposed the closed-form solutions to estimate the displacements and stresses owing to various loading types of an axially loaded plumb pile. The loading types are composed of an embedded point load for an end-bearing plumb pile, and a uniform skin friction, a linear variation of skin friction, a non-linear variation of skin friction for a friction plumb pile. Nevertheless, in order to resist the significant lateral forces, piles are designed with an inclination to the vertical. On this subject, Ramiah and Chickanagappa [11] presented the vertical stress solutions for piles battered into an isotropic medium, due to an embedded point load, a uniform skin friction, and a linear variation of skin friction. Recently, Wang et al. [14] yielded the analytical solutions for calculating the vertical surface displacements resulting from the loading types of an end bearing, a uniform skin friction, and a linear variation of skin friction for batter piles driven into cross-anisotropic media. They developed the solutions by integrating the embedded point load solutions of displacements for a cross-anisotropic half-space, which were provided by Wang and Liao [15]. Their solutions indicated that the surface displacements in cross-anisotropic media were governed by the type and degree of material anisotropy, angle of inclination, and distinct loading types. However, our investigations reveal that there are no stress solutions existing for piles battered into cross-anisotropic media. In the derivations of present solutions, we assume that the media are homogeneous, linearly elastic, and the planes of cross-anisotropy are parallel to the ground surface. In addition, the influence of batter pile’s diameter ($D$) is not considered, which means that the distribution of pressure along the length of a pile is a function of the length ($L$) to diameter ($D$) ratio [11, 14]. Hence, this article inherits the aforementioned assumptions, and the analytical solutions of vertical stress induced by the following three loading cases for batter piles pushed into cross-anisotropic media are presented:

Case A: Point load case: a point load $P$ (force) acts along the oblique local axis, $z$, at the pile length $L$.

Case B: Uniform skin friction case: a total load $Q$ (force per unit length) applies along the oblique local axis, $z$, in uniform distribution ($Q(z) = (P/L)$) from the ground surface to the pile length $L$. 

**Case C**: Linear variation of skin friction case: a total load \( Q \) (force per unit length) applies along the oblique local axis, \( z \), in increments varying linearly with depth \((Q(z) = (2P/L^2)z)\), from zero at the ground surface to a maximum at the pile length \( L \).

The proposed solutions can be achieved by suitable integrations of Wang and Liao’s point load solutions \[15\]. Furthermore, an example is illustrated to clarify the impact of the type and degree of soil anisotropy, pile inclination, and loading types on vertical stresses in the isotropic/cross-anisotropic media.

### VERTICAL STRESS DUE TO AN EMBEDDED HORIZONTAL/VERTICAL POINT LOAD

Figure 1 depicts a cross-anisotropic half-space due to an embedded horizontal/vertical point load \( P_x/P_z \), at depth \( h \), in the global co-ordinate system \((X, Y, Z)\). The cross-anisotropic planes are assumed parallel to the horizontal ground surface. The detailed procedures for deriving the solutions of three normal and three shear stresses can be referred to Wang and Liao \[15\]. In this work, the most interesting quantity, namely, the vertical stress \((\sigma_{zz}^P)\) induced by \( P_x/P_z \) is considered and presented as

\[
\sigma_{zz}^P = \frac{P_x}{4\pi} \left[ (A_{13} - u_1m_1A_{33}) \left( \frac{k}{m_1} p_{s11} - T_1 p_{s1a} + T_2 p_{s1b} \right) 
- (A_{13} - u_2m_2A_{33}) \left( \frac{k}{m_2} p_{s12} - T_3 p_{s1c} + T_4 p_{s1d} \right) \right] 
+ \frac{P_z}{4\pi} \left[ (A_{13} - u_1m_1A_{33})(kp_{s21} + T_1 m_1 p_{s2a} - T_2 m_2 p_{s2b}) 
- (A_{13} - u_2m_2A_{33})(kp_{s22} + T_3 m_1 p_{s2c} - T_4 m_2 p_{s2d}) \right] \tag{1}
\]

![Figure 1. The cross-anisotropic media due to an embedded horizontal/vertical point load \( P_x/P_z \) in the global co-ordinate system.](image-url)
Table I. The relationships of $Z_i$ ($i = 1, 2, a, b, c, d$) and $u_m Z + u_n h$ ($m, n = 1, 2$).

<table>
<thead>
<tr>
<th>$Z_i$ ($i = 1, 2, a, b, c, d$)</th>
<th>$u_m Z + u_n h$ ($m, n = 1, 2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z_1 = u_1 (Z - h)$</td>
<td>$u_n = -u_m = -u_1$</td>
</tr>
<tr>
<td>$Z_2 = u_2 (Z - h)$</td>
<td>$u_n = -u_m = -u_2$</td>
</tr>
<tr>
<td>$Z_d = u_1 (Z + h)$</td>
<td>$u_n = u_m = u_1$</td>
</tr>
<tr>
<td>$Z_d = u_2 (Z + h)$</td>
<td>$u_n = u_m = u_2$</td>
</tr>
<tr>
<td>$Z_b = u_1 Z + u_2 h$</td>
<td>$u_m = u_1, u_n = u_2$</td>
</tr>
<tr>
<td>$Z_c = u_2 Z + u_1 h$</td>
<td>$u_m = u_2, u_n = u_1$</td>
</tr>
</tbody>
</table>

where the superscript $p$ indicates the vertical stress subjected to a point load in a global co-ordinate system, and

\[
k = \frac{A_{13} + A_{44}}{A_{33} A_{44} (u_1^2 - u_2^2)}, \quad m_j = \frac{(A_{13} + A_{44})u_j}{A_{33}u_j^2 - A_{44}} = \frac{A_{11} - A_{44}u_j^2}{(A_{13} + A_{44})u_j} \quad (j = 1, 2)
\]

\[
T_1 = k \frac{u_1 + u_2}{m_1 u_2 - u_1}, \quad T_2 = k \frac{2u_1 (u_2 + m_2)}{m_1 (u_2 - u_1)(u_1 + m_1)}, \quad T_3 = k \frac{2u_2 (u_1 + m_1)}{m_1 (u_2 - u_1)(u_2 + m_2)}
\]

\[
T_4 = k \frac{u_1 + u_2}{m_2 u_2 - u_1}, \quad p_{3i} = \frac{X}{R_i^3}, \quad p_{3i} = \frac{Z_i}{R_i^3}, \quad R_i = \sqrt{X^2 + Y^2 + Z_i^2} \quad (i = 1, 2, a, b, c, d)
\]

and $Z_i$ has three distinct forms: (1) $Z_1 = u_1 (Z - h)$, $Z_2 = u_2 (Z - h)$; (2) $Z_d = u_1 (Z + h)$, $Z_d = u_2 (Z + h)$; and (3) $Z_b = u_1 Z + u_2 h$, $Z_c = u_1 h + u_2 Z$. Now, $p_{3i}$ and $p_{3i}$ ($i = 1, 2, a, b, c, d$) in Equation (1) are the stress elementary functions for vertical stress at any point resulting from a horizontal and vertical subsurface loading in a cross-anisotropic half-space. In addition, if $Z_i$ ($i = 1, 2, a, b, c, d$) = $u_m Z + u_n h$ ($m, n = 1, 2$), that is, dealing with the general form, $u_m Z + u_n h$, is easier than dealing with the three forms of $Z_i$. Table I expresses the relationships of $Z_i$ ($i = 1, 2, a, b, c, d$) and $u_m Z + u_n h$ ($m, n = 1, 2$).

$A_{11}$, $A_{13}$, $A_{33}$, $A_{44}$ are the elastic moduli or elasticity constants of the medium, and can be represented as [14]

\[
A_{11} = \frac{E \left(1 - \frac{E'}{E^2} v^2\right)}{(1 + v)(1 - \frac{2E}{E^2} v^2)}, \quad A_{13} = \frac{E' v}{1 - \frac{2E}{E^2} v^2}, \quad A_{33} = \frac{E' (1 - v)}{1 - \frac{2E}{E^2} v^2}, \quad A_{44} = G'
\]

For a cross-anisotropic medium, the five engineering elastic constants, $E$, $E'$, $v$, $v'$, and $G'$ are defined as:

1. $E$ is Young’s modulus in the horizontal direction.
2. $E'$ is Young’s modulus in the vertical direction.
3. $v$ is the Poisson ratio for the effect of horizontal stress on complementary horizontal strain.
4. $v'$ is the Poisson ratio for the effect of vertical stress on horizontal strain.
5. $G'$ is the shear modulus in the vertical plane.
\( u_1 \) and \( u_2 \) are the roots of the following characteristic equation:

\[
u^4 - su^2 + q = 0 \tag{3}
\]

where

\[
s = \frac{A_{11}A_{33} - A_{13}(A_{13} + 2A_{44})}{A_{33}A_{44}}, \quad q = \frac{A_{11}}{A_{33}}
\]

There are three categories of the characteristic roots, \( u_1 \) and \( u_2 \) as:

- **Case 1**: \( u_{1,2} = \pm \sqrt{\frac{1}{2}[s \pm \sqrt{(s^2 - 4q)}]} \) are two real distinct roots when \( s^2 - 4q > 0 \);
- **Case 2**: \( u_{1,2} = \pm \sqrt{s/2}, \pm \sqrt{s/2} \) are double equal real roots when \( s^2 - 4q = 0 \);
- **Case 3**: \( u_1 = \frac{1}{2}\sqrt{(s + 2\sqrt{q}) - i\frac{1}{2}\sqrt{-(s + 2\sqrt{q})}} = \gamma - i\delta, \quad u_2 = \gamma + i\delta \) are two complex conjugate roots where \( \gamma \) cannot be equal to zero when \( s^2 - 4q < 0 \).

Suppose a horizontal point load \( P_x \), and a vertical one \( P_z \), apply at a buried depth \( L \) (the pile length), then, the vertical stress for an end-bearing pile in the cross-anisotropic half-space can be acquired by substituting \( h \) by \( L \) in the aforementioned \( Z_i \) (\( i = 1, 2, a, b, c, d \)).

According to Equation (1), the vertical stresses due to an end-bearing point load (Case A), a uniform skin friction (Case B), and a linear variation of skin friction (Case C) for piles battered into a cross-anisotropic medium can be derived and presented as follows:

**CASE A: VERTICAL STRESS DUE TO AN END-BEARING BATTER PILE**

In this article, an oblique local co-ordinate system \((x, y, z)\), as shown in Figure 2, is employed to generate the vertical stress solutions induced by batter piles. Figure 2 displays an inclined point load, \( P \), at an angle, \( \alpha \), with respect to the vertical, along the \( z \)-axis of an oblique local co-ordinate system. The point load, \( P \), can be decomposed into a horizontal component, \( P \sin \alpha \), and a vertical...
Figure 3. Three distinct loading types for a batter pile driven in cross-anisotropic media: (a) point load case (Case A); (b) uniform skin friction case (Case B); and (c) linear variation of skin friction case (Case C).
component, $P \cos \alpha$. That is, the vertical stress due to these components corresponding to the local co-ordinate system can be obtained by replacing the point load solution (Equation (1)) as

$$\sigma_{zz}^p = \frac{P \sin \alpha}{4\pi} \left[ (A_{13} - u_1m_1A_{33}) \left( \frac{k}{m_1} p_{s11}' - T_1 p_{s1a}' + T_2 p_{s1b}' \right) \right]$$

$$+ (A_{13} - u_2m_2A_{33}) \left( \frac{k}{m_2} p_{s12}' - T_3 p_{s1c}' + T_4 p_{s1d}' \right)$$

where in Equation (4), $p_{s1i}'$ and $p_{s2i}'$ ($i = 1, 2, a, b, c, d$) are defined as the new stress elementary functions, and they can be modified from $p_{s1i}$ and $p_{s2i}$ in Equation (1), from the original global co-ordinate system $(X, Y, Z)$ to the new local one $(x, y, z)$ as

$$p_{s1i}' = \frac{x + (z-h) \sin \alpha}{R_i^3}$$

$$p_{s2i}' = \frac{z_i'}{R_i^3}$$

where $R_i' = \sqrt{(x + (z-h) \sin \alpha)^2 + y^2 + z_i'^2}$, $z_i' = (u_m z + u_n h) \cos \alpha$ ($i = 1, 2, a, b, c, d$; $m, n = 1, 2$).

Figure 3(a) plots an end-bearing batter pile driven at an angle, $\alpha$ (with respect to the vertical), in the oblique local co-ordinate system, into a cross-anisotropic medium. The induced vertical stress can be calculated using Equations (4)–(6) by substituting $h$ by $L$. The yielded solutions in a limiting approach are identical to the isotropic solutions of Ramiah and Chickanagappa [11] considering the pile inclination. Moreover, the present solutions are in good agreement with Wang’s cross-anisotropic solutions [13] when batter angle equals to 0 (i.e. $\alpha = 0^\circ$).

CASE B: VERTICAL STRESS DUE TO A UNIFORM SKIN FRICTION BATTER PILE

Figure 3(b) shows a total load $Q$ (force per unit length) acting along the oblique local $z$-axis of a batter pile in uniform increments from the ground surface to the pile length $L$. Taking an infinitesimal element $dh$ along the oblique local $z$-axis, the total load can be divided into a finite number of elementary forces as

$$dQ = \left( \frac{P}{L} \right) \, dh$$

The exact solution for vertical stress due to a uniform skin friction of a batter pile in the cross-anisotropic media is similar to Equation (4) except for the new stress elementary functions,
functions, \( p'_{s1i} \) (Equation (5)) and \( p'_{s2i} \) (Equation (6)), which are, respectively, replaced by the stress integral functions, \( r_{s1i} \) and \( r_{s2i} \) \((i = 1, 2, a, b, c, d)\) as

\[
\sigma'_{zz}(\text{oblique}) = \frac{P\sin \alpha}{4\pi} \left[ \left( A_{13} - u_1m_1A_{33} \right) \left( \frac{k}{m_1}r_{s11} - T_1r_{s1a} + T_2r_{s1b} \right) \right. \\
- \left. \left( A_{13} - u_2m_2A_{33} \right) \left( \frac{k}{m_2}r_{s12} - T_3r_{s1c} + T_4r_{s1d} \right) \right]
\]

\[
+ \frac{P\cos \alpha}{4\pi} \left[ \left( A_{13} - u_1m_1A_{33} \right) (kr_{s21} + T_1m_1r_{s2a} - T_2m_2r_{s2b}) \right.
\]

\[
- \left. \left( A_{13} - u_2m_2A_{33} \right) (kr_{s22} + T_3m_1r_{s2c} - T_4m_2r_{s2d}) \right] \tag{8}
\]

The stress integral functions, \( r_{s1i} \) and \( r_{s2i} \), in Equation (8) are obtained by integrating \( h \) of the new stress elementary functions, \( p'_{s1i} \) (Equation (5)) and \( p'_{s2i} \) (Equation (6)), from 0 to \( L \) as

\[
\sigma'_{zz}(\text{oblique}) = \int_0^L \sigma'_{zz}(\text{oblique}) \, dQ = \int_0^L \sigma'_{zz}(\text{oblique}) \left( \frac{P}{L} \right) \, dh \tag{9}
\]

where the superscript \( r \) means the vertical stress owing to a uniform skin friction for a batter pile. However, as aforementioned, the variable \( h \) that associates with \( Z_i \) has three distinct forms, and hence, entire presentations for all types would be lengthy. Thus, only \( r_{s1i} \) and \( r_{s2i} \) in terms of \( u_m \) and \( u_n \) \((m, n = 1, 2)\) are expressed as

\[
r_{s1i} = -\frac{\sqrt{2}}{E_{mn}L} \left\{ \begin{array}{c}
y^2 \sin \alpha + u_mz[u_nx + (u_m + u_n)z \sin \alpha] \cos^2 \alpha \\
y^2 \sin \alpha + (u_mz + u_nL)[u_nx + (u_m + u_n)z \sin \alpha] \cos^2 \alpha \\
B_m \\
C_{mn} \end{array} \right\} \tag{10}
\]

\[
r_{s2i} = \frac{\sqrt{2}\cos \alpha}{E_{mn}L} \left\{ \begin{array}{c}
(x + z \sin \alpha)[u_mz \sin \alpha + u_n(x + z \sin \alpha)] \\
u_m(x^2 + y^2) + (u_m + u_n)xz \sin \alpha + (u_nx + z \sin \alpha)(z - L) \sin \alpha \\
B_m \\
C_{mn} \end{array} \right\} \tag{11}
\]

where \( A_n, B_m, C_{mn}, \) and \( E_{mn} \) \((m, n = 1, 2)\) are shown below:

\[
A_n = \sqrt{2}\sqrt{\sin^2 \alpha + (u_n \cos \alpha)^2}
\]

\[
B_m = \sqrt{2}\sqrt{(x + z \sin \alpha)^2 + y^2 + (u_mz \cos \alpha)^2}
\]

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VERTICAL STRESS DISTRIBUTIONS AROUND BATTER PILES

\[ C_{mn} = \sqrt{2} \sqrt{[x + (z - L) \sin \alpha]^2 + y^2 + [(u_m z + u_n L) \cos \alpha]^2} \]

\[ E_{mn} = [u_n x + (u_m + u_n) z \sin \alpha]^2 \cos^2 \alpha + \frac{1}{2} A_n^2 y^2 \]

The generated solutions in limited forms are totally the same as Ramiah and Chickanagappa’s solutions [11] for a uniformly distributed pile pushed into an isotropic medium. Additionally, the present formulae are equivalent to Wang’s solutions [13] for a uniform skin friction plumb pile \((\alpha = 0^\circ)\) driven into a cross-anisotropic geomaterial.

CASE C: VERTICAL STRESS DUE TO A LINEAR VARIATION OF SKIN FRICTION BATTER PILE

Figure 3(c) depicts a total load \(Q\) (force per unit length) applying along the axis of the batter pile \((z\)-axis) in increments varying linearly with depth, from zero at the ground surface to maximum at the pile length \(L\). The load that acts over an infinitesimal element \(dh\) is presented as

\[ dQ = 2P \left( \frac{h}{L^2} \right) dh \] (12)

Substituting Equation (12) into \(p_{s1l}^l\) (Equation (5)) and \(p_{s2l}^l\) (Equation (6)), and integrating \(h\) between the limits 0 and \(L\) as

\[ \sigma_{zz}^l(\text{oblique}) = \int_0^L \sigma_{zz}^p(\text{oblique}) dQ = \int_0^L \sigma_{zz}^p(\text{oblique}) \left( \frac{2Ph}{L^2} \right) dh \] (13)

where the superscript \(l\) means the vertical stress resulting from a linear variation of skin friction for a batter pile. The complete vertical stress solution and new stress integral functions, \(l_{s1l}\) and \(l_{s2l}\) \((i = 1, 2, a, b, c, d)\) are written as

\[ \sigma_{zz}^l(\text{oblique}) = \frac{P \sin \alpha}{4\pi} \left[ (A_{13} - u_1 m_1 A_{33}) \left( \frac{k}{m_1} l_{s11} - T_1 l_{s1a} + T_2 l_{s1b} \right) \right. \\
- \left. (A_{13} - u_2 m_2 A_{33}) \left( \frac{k}{m_2} l_{s12} - T_3 l_{s1c} + T_4 l_{s1d} \right) \right] \\
+ \frac{P \cos \alpha}{4\pi} \left[ (A_{13} - u_1 m_1 A_{33})(kl_{s21} + T_1 m_1 l_{s2a} - T_2 m_2 l_{s2b}) \right. \\
\left. -(A_{13} - u_2 m_2 A_{33})(kl_{s22} + T_3 m_1 l_{s2c} - T_4 m_2 l_{s2d}) \right] \] (14)

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where

\[
l_{si} = -\frac{1}{\sqrt{2A_n^2 B_m C_{mn} E_{mn} L^2}}
\]

\[
4A_n u_n^2 x \cos 2\alpha \\
\begin{aligned}
&\left( B_m - C_{mn} \right) (x^2 + y^2 + u_n^2 z^2) \\
&+ B_m L u_m u_n z
\end{aligned}
\]

\[
+ u_n x \\
A_n \left\{ 4(B_m - C_{mn}) u_n (x^2 + y^2) \\
+ B_m L [ -4u_n + 3u_n (u_n^2 - 1)] z \\
+ (B_m - C_{mn}) [2u_n + 3(u_n^2 + 1)u_n] z^2 \\
- 2B_m C_{mn} (u_m + u_n) z \ln |F_{mn}| \right\}
\]

\[
+ \sin \alpha \\
\begin{aligned}
&\left( B_m - C_{mn} \right) u_n [(u_m + 3u_n) x^2 + (u_m + u_n) y^2] z \\
&+ B_m L (u_m + u_n) [ -2u_n + u_m (u_n^2 - 1)] z^2 \\
&+ (B_m - C_{mn}) (u_m^2 + 1) u_n (u_m + u_n) z^3 \\
- B_m C_{mn} [ 6y^2 + 2u_n^2 (x^2 + y^2) + (u_m + u_n)^2 z^2 ] \ln |F_{mn}| \right\}
\]

\[
\times \\
\begin{aligned}
&\left( B_m - C_{mn} \right) u_n [(u_m + 3u_n) x^2 + (u_m + u_n) y^2] z \\
&+ B_m L (u_m + u_n) [ -2u_n + (u_m^2 - 3)u_n] z \\
+ 2B_m C_{mn} (u_m + u_n) \ln |F_{mn}| \right\}
\]

\[
+ z \cos 4\alpha \\
\begin{aligned}
&\left( B_m - C_{mn} \right) u_n [(u_m + 3u_n) x^2 + (u_m + u_n) y^2] z \\
&+ B_m L (u_m + u_n) [ -2u_n + (u_m^2 - 3)u_n] z \\
+ 2B_m C_{mn} (u_m + u_n) \ln |F_{mn}| \right\}
\]

\[
+ \sin 4\alpha \\
\begin{aligned}
&\left( B_m - C_{mn} \right) u_n [(u_m + 3u_n) x^2 + (u_m + u_n) y^2] z \\
&+ B_m L (u_m + u_n) [ -2u_n + (u_m^2 - 3)u_n] z \\
+ 2B_m C_{mn} (u_m + u_n) \ln |F_{mn}| \right\}
\]

\[
+ 2 \sin 3\alpha \\
\begin{aligned}
&\left( B_m - C_{mn} \right) u_n [(u_m + 3u_n) x^2 + (u_m + u_n) y^2] z \\
&+ B_m L (u_m + u_n) [ -2u_n + (u_m^2 - 3)u_n] z \\
+ 2B_m C_{mn} (u_m + u_n) \ln |F_{mn}| \right\}
\]
\[ l_{2\ell} = -\frac{\cos x}{\sqrt{2}A_n^3 B_m C_{mn} E_{mn} L^2} \]

\[
\begin{aligned}
A_n &\left\{ \begin{array}{l}
4B_m L u_n [(u_n^2 - 1)x^2 + (u_n^2 + 1)y^2] \\
+4(B_m - C_{mn}) [(u_m + 3u_n)x^2 + (u_m + u_n)y^2]z \\
+ B_m L (u_m + u_n) (u_n^2 + 2u_m u_n - 3)z^2 \\
+ (B_m - C_{mn}) (u_n^2 + 3) (u_m + u_n) z^3 \\
\end{array} \right\} \\
\end{aligned}
\]

\[
\begin{aligned}
\cos 2x &\left\{ \begin{array}{l}
B_m L u_n [(u_n^2 + 1)x^2 + (u_n^2 - 1)y^2] \\
- (B_m - C_{mn}) [(u_m + 3u_n)x^2 + (u_m + u_n)y^2]z \\
+ B_m L (u_m + u_n) z^2 \\
- (B_m - C_{mn}) (u_m + u_n) z^3 \\
\end{array} \right\} \\
\end{aligned}
\]

\[
\begin{aligned}
+4 \cos 2x &\left\{ \begin{array}{l}
+ B_m L (u_m + u_n) z^2 \\
+ A_n (B_m - C_{mn}) (u_n^2 - 1)z \\
+ B_m C_{mn} u_n (u_m + u_n) \ln |F_{mn}| \\
\end{array} \right\} \\
\end{aligned}
\]

\[
\sin x \left\{ \begin{array}{l}
4(B_m - C_{mn}) u_n (x^2 + y^2) \\
+ B_m L [2u_n (u_n^2 - 3) + 3u_m (u_n^2 - 1)]z \\
+ (B_m - C_{mn}) [6u_m + (u_m^2 + 9) u_n] z^2 \\
+ 2B_m C_{mn} u_n^2 (u_m + u_n) \ln |F_{mn}| \\
\end{array} \right\} \\
\end{aligned}
\]

\[
\begin{aligned}
\sin x &\left\{ \begin{array}{l}
+ B_m L [u_n (3u_n^2 + 1) + 2u_n (u_n^2 + 1)] \\
+ A_n (B_m - C_{mn}) [(u_n^2 - 3) u_n - 2u_m] z \\
+ 2B_m C_{mn} u_n^2 (u_m + u_n) \ln |F_{mn}| \\
\end{array} \right\} \\
\end{aligned}
\]

\[
\begin{aligned}
2x &\left\{ \begin{array}{l}
\frac{4(B_m - C_{mn}) u_n (x^2 + y^2)}{A_n B_m + D_{mn}} \\
+ B_m L [2u_n (u_n^2 - 3) + 3u_m (u_n^2 - 1)]z \\
+ (B_m - C_{mn}) [6u_m + (u_m^2 + 9) u_n] z^2 \\
+ 2B_m C_{mn} u_n^2 (u_m + u_n) \ln |F_{mn}| \\
\end{array} \right\} \\
\end{aligned}
\]

\[
\begin{aligned}
\sin 3x &\left\{ \begin{array}{l}
\frac{A_n B_m L [u_n (3u_n^2 + 1) + 2u_n (u_n^2 + 1)]}{A_n (C_{mn} + LA_n) + D_{mn}} \\
+ A_n (B_m - C_{mn}) [(u_n^2 - 3) u_n - 2u_m] z \\
+ 2B_m C_{mn} u_n^2 (u_m + u_n) \ln |F_{mn}| \\
\end{array} \right\} \\
\end{aligned}
\]

and

\[ D_{mn} = -2[x \sin x + z (\sin^2 x - u_m u_n \cos^2 x)] \]

\[ F_{mn} = \frac{A_n B_m + D_{mn}}{A_n (C_{mn} + LA_n) + D_{mn}} \]

The proposed solutions for a linear variation of skin friction in a limiting procedure are in perfect agreement with Ramiah and Chickanagappa’s solution [11] when the medium is isotropic.
Table II. Solutions of vertical stress for present loading Cases A–C.

<table>
<thead>
<tr>
<th>Type of loading</th>
<th>Vertical stress solutions</th>
<th>Stress elementary/integral functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case A</td>
<td>Equation (4)</td>
<td>Equations (5) and (6) by substituting $h$ by $L$</td>
</tr>
<tr>
<td>Case B</td>
<td>Equation (8)</td>
<td>Equations (10) and (11)</td>
</tr>
<tr>
<td>Case C</td>
<td>Equation (14)</td>
<td>Equations (15) and (16)</td>
</tr>
</tbody>
</table>

Furthermore, the present batter pile solutions are conformed to Wang’s solutions [13] for plumb piles driven in cross-anisotropic media. Regarding the vertical stresses due to any conceivable loading the superposition principle can be employed. For example, suppose a loading case of a friction load varying linearly from a maximum at the ground surface to zero at the pile length $L$, the solution, $\sigma_{zz}^{d}(\text{oblique})$ can be simply acquired as

$$
\sigma_{zz}^{d}(\text{oblique}) = 2*\sigma_{zz}^{r}(\text{oblique}) - \sigma_{zz}^{l}(\text{oblique})
$$

Table II exhibits the complete expressions of vertical stress and related stress elementary/integral functions for present loading Cases A–C.

ILLUSTRATIVE EXAMPLE

This section presents an example to examine the yielded solutions and elucidate the effect of the type and degree of material anisotropy, pile inclination angle, and three distinct loading types (Cases A–C) on the vertical stresses for piles battered into isotropic/cross-anisotropic media. Practically, pile batters range from 1:12 (horizontal:vertical) to 5:12 [16]. Therefore, piles inclined at an angle ($\alpha$) varying between 0 and 30° with respect to the vertical are adopted in this parametic study. For typical ranges of cross-anisotropic parameters, Gazetas [17] summarized several experimental data regarding deformational cross-anisotropy of clays and sands. He suggested the anisotropic ratio $E/E'$ for clays ranging from 0.6 to 4, and as low as 0.2 for sands. Hence, the range of ratio $E/E'$ is 1.35–2.37, and that of ratio $G'/E'$ is 0.23–0.44 [18–21], and we assume the ratio $v/v'$ is 0.75–1.5.

However, in this study, we adopt a very famous over-consolidated fissured clay—London clay as our demonstration. The characteristics of London clay are, respectively, discussed briefly by Hight et al. [22], Gasparre et al. [23, 24], Nishimura et al. [25], and Wongsaroj et al. [26]. Consequently, the London clay anisotropic degree including the ratios $E/E'$, $G'/E'$, and $v/v'$, are interpreted to investigate their effect on the vertical stresses. Isotropic (Soil 1) and cross-anisotropic soils (Soils 2–4) are chosen as the constituted foundation geomaterials. Their elastic properties are listed in Table III. In Table III, $E$ and $v$ are, respectively, 50 MPa [27] and 0.3 [28].

A Mathematica® program based on Equations (4)–(6) (for a point load), Equations (8), (10) and (11) (for a uniform skin friction), and Equations (14)–(16) (for a linear variation of skin friction), is written to compute the vertical stresses. This program can estimate the vertical stress at any point in the isotropic/cross-anisotropic media. In this section, we choose the most complicated but important loading case—a linear variation of skin friction (Case C) to exhibit the effect of soil anisotropy and pile inclination on the non-dimensional vertical stress for batter piles. The non-dimensional vertical stress for Soils 1–4 along the inclined line of action ($x=y=0$) of a linear variation of skin friction ($\sigma_{zz}^{l}(\text{oblique}) * L^2/P$) vs the non-dimensional ratio $z/L$ are given...
in Figures 4 and 5. As for the effect of three loading types on the non-dimensional vertical stresses, we select Soil 2 \( (E/E' = 2.37, G'/E' = 0.385, v/v' = 1.0, \text{ Table III}), \) as shown in Figure 6, to clarify \( \sigma_{zz(\text{oblique})}^p * L^2/P \) (Case A), \( \sigma_{zz(\text{oblique})}^t * L^2/P \) (Case B), \( \sigma_{zz(\text{oblique})}^l * L^2/P \) (Case C) vs \( z/L \).

First, in order to check the accuracy of generated solutions, comparisons with the isotropic solutions of Ramiah and Chickanagappa \([11]\) are verified by a limiting approach. Then, the present solutions are also compared with the cross-anisotropic solutions of Wang \([13]\) when \( \alpha = 0^\circ \). Results reveal that the proposed solutions are exactly identical to the isotropic solutions of Ramiah and Chickanagappa’s \([11]\) for batter piles and the cross-anisotropic solutions of Wang \([13]\) for plumb piles.

Figure 4 displays the effect of ratios \( E/E', G'/E', \) and \( v/v' \) (soil anisotropy) on the non-dimensional vertical stress \((\sigma_{zz(\text{oblique})}^p * L^2/P) \) vs \( z/L \) from 1 to 3, owing to a linear variation of skin friction pile with the inclination angle \( \alpha = 0^\circ \) (plumb pile case, Figure 4(a)), \( 10^\circ \) (Figure 4(b)), \( 20^\circ \) (Figure 4(c)), and \( 30^\circ \) (Figure 4(d)). Figure 4(a) reveals that \( \sigma_{zz(\text{oblique})}^p * L^2/P \) (1) decreases with the increase in \( z/L \) from 1 to 3; (2) slightly increases with the increase of \( E/E' \) from 1.0 to 2.37 (Soils 1 and 2); (3) increases with the decrease of \( G'/E' \) from 0.385 to 0.23 (Soils 1 and 3); and (4) is little affected by the ratio \( v/v' \) (Soils 1 and 4). Especially, the decrease of ratio \( G'/E' \) does have a remarkable influence on the vertical stress. Regarding the trends of Figures 4(b)–(d), they are similar to those of Figure 4(a); however, with the increase of batter angle (from 0 to 30\(^{\circ}\)), the magnitudes of \( \sigma_{zz(\text{oblique})}^p * L^2/P \) are gradually insensitive to the soil anisotropy. Figures 5(a)–(d) demonstrate the effect of pile inclination with various batter angles of 0, 10, 20, and 30\(^{\circ}\) on \( \sigma_{zz(\text{oblique})}^l * L^2/P \) due to a linear variation of skin friction pile, respectively, driven in Soils 1–4. It can be discovered from Figure 5(a) that the calculated result from \( \alpha = 10^\circ \) is almost the same with that from \( \alpha = 0^\circ \), meaning that within the small range of \( \alpha \) (i.e. 0–10\(^{\circ}\)), pile inclination factor has little impact on \( \sigma_{zz(\text{oblique})}^l * L^2/P \) for a linear variation of skin friction pile battered into an isotropic soil (Soil 1). Nevertheless, the effect of pile inclination on \( \sigma_{zz(\text{oblique})}^l * L^2/P \) becomes explicit for cross-anisotropic soils, in particular for Soil 3 with \( E/E' = 1.0, G'/E' = 0.23, \) and \( v/v' = 1.0. \) Furthermore, increasing \( \alpha \) would decisively decrease the magnitude of \( \sigma_{zz(\text{oblique})}^l * L^2/P; \) that is the calculations by present solutions for batter piles predict lower vertical stress than those by Ramiah and Chickanagappa’s isotropic solutions \([11]\), and Wang’s cross-anisotropic solutions \([13]\).

Eventually, the magnitudes of vertical stress induced by a point load \((\sigma_{zz(\text{oblique})}^p * L^2/P)\), a uniform skin friction \((\sigma_{zz(\text{oblique})}^t * L^2/P)\), and a linear variation of skin friction \((\sigma_{zz(\text{oblique})}^l * L^2/P)\) might be different. Consequently, Figure 6 presents the effect of three distinct loading types (Cases A–C) on the non-dimensional vertical stresses for cross-anisotropic Soil 2 \((E/E' = 2.37,\)}
Figure 4. Effect of soil anisotropy on non-dimensional vertical stress due to a linear variation of skin friction pile in the cross-anisotropic media when: (a) $\alpha=0^\circ$; (b) $\alpha=10^\circ$; (c) $\alpha=20^\circ$; and (d) $\alpha=30^\circ$.
Figure 5. Effect of pile inclination on non-dimensional vertical stress due to a linear variation of skin friction pile when \( z = 0^\circ, 10^\circ, 20^\circ, \) and \( 30^\circ \) for: (a) Soil 1; (b) Soil 2; (c) Soil 3; and (d) Soil 4.

partially since the gravity effects would produce a downward flow of the soil mass to eliminate them [13, 16]. Moreover, based on the parametric study results (Figures 4–6), the magnitudes and distributions of non-dimensional vertical stresses profoundly rely on the type and degree of soil anisotropy \( (E/E', G'/E', \nu/\nu') \), pile inclination \( (z) \), and three distinct loading types (Cases A–C). Hence, the pile inclination factor should be considered when end-bearing or skin friction piles battered into the cross-anisotropic media.
Figure 6. Effect of loading types on non-dimensional vertical stresses due to a point load, a uniform skin friction, and a linear variation of skin friction in the cross-anisotropic Soil 2, when: (a) $\alpha = 0^\circ$; (b) $\alpha = 10^\circ$; (c) $\alpha = 20^\circ$; and (d) $\alpha = 30^\circ$.

CONCLUSIONS

This article presents the closed-form solutions to estimate the vertical stresses resulting from three distinct loading types for batter piles driven in the cross-anisotropic media. The loading types
are an embedded point load for an end-bearing pile, uniform skin friction, and linear variation of skin friction for a friction pile. The deriving processes are based on Wang and Liao’s point solutions [15], which can be expressed in terms of two new stress elementary functions in an oblique local co-ordinate system for vertical stress due to a horizontal and vertical embedded point load in a cross-anisotropic half-space. Then, the new stress elementary functions are integrated to generate the solutions of vertical stress induced by a uniform and a linear variation of skin friction for pile battered into cross-anisotropic media. An example utilizes the typical range of over- consolidated London clay, which is demonstrated to analyze the effect of the type and degree of soil anisotropy ($E/E', G'/E'$, $v/v'$), pile inclination ($\alpha$), and loading types (a point load (Case A), a uniform skin friction (Case B), and a linear variation of skin friction (Case C)) on the vertical stresses.

The following conclusions are grounded upon the results of a parametric study for the illustration example:

1. The derived solutions have been verified with the isotropic solutions of Ramiah and Chickanagappa [11] by a limiting approach. In addition, the present solutions are exactly identical to the cross-anisotropic solutions of Wang [13] when batter angle is $\alpha=0^\circ$.
2. The effect of anisotropic ratios $E/E'$, $G'/E'$, and $v/v'$ on the non-dimensional vertical stress ($\sigma^{I}_{zz(\text{oblique})} \times L^2 / P$) suggests that the stress (1) decreases with the increase of $z/L$ from 1 to 3; (2) slightly increases with the increase of $E/E'$ from 1.0 to 2.37 (Soils 1 and 2); (3) increases with the decrease of $G'/E'$ from 0.385 to 0.23 (Soils 1 and 3); and (4) is nearly unaffected by the ratio $v/v'$ (Soils 1 and 4). Particularly, the decrease of ratio $G'/E'$ does have a great influence on the stress.
3. Within a small range of $\alpha$ (i.e. $0^\circ$–$10^\circ$), the pile inclination factor has little impact on the induced stress ($\sigma^{I}_{zz(\text{oblique})} \times L^2 / P$) for piles pushed into an isotropic soil (Soil 1). However, the effect of battle angle on the vertical stress becomes explicit for cross-anisotropic soils, especially for Soil 3 ($E/E'=1.0, G'/E'=0.23, v/v'=1.0$). Additionally, the calculations by present solutions for batter piles predict lower vertical stress than those by using Ramiah and Chickanagappa’s isotropic solutions [11], and Wang’s cross-anisotropic solutions [13].
4. The effect of three distinct loading types on the non-dimensional vertical stresses for Soil 2 ($E/E'=2.37, G'/E'=0.385, v/v'=1.0$) reveals the order-induced stresses that follow: the point load case $>$ the linear variation of skin friction case $>$ the uniform skin friction case.
5. The estimation of vertical stresses owing to the proposed loading types in an isotropic/cross-anisotropic medium is very efficient and correct since the presentation of the closed-form solutions is concise and systematized. Moreover, the vertical stresses caused by any compound form of loading can be solved by using the principle of superposition.

The present vertical stress solutions as well as the surface displacement solutions [14] could offer practical engineers a complete and useful reference when piles are driven in an isotropic/cross-anisotropic earth mass. Furthermore, a parametric analysis demonstrates that the vertical stresses with the consideration of pile batter are considerably different from those calculated from plumb piles. It is again interpreted that the pile inclination factor must be studied when piles are designed to resist both the actions of axial and lateral load in the cross-anisotropic media.
APPENDIX A: NOMENCLATURE

\( A_{11}, A_{13}, A_{33}, A_{44} \) elastic moduli or elasticity constants
\( dh \) infinitesimal element along the oblique local \( z \)-axis
\( E \) Young’s modulus in the horizontal direction
\( E' \) Young’s modulus in the vertical direction
\( G' \) shear modulus in the vertical plane
\( i \) complex number \((=\sqrt{-1})\)
\( k, m_1, m_2, T_1, T_2, T_3, T_4 \) coefficients
\( L \) the pile length
\( l_{s1i} - l_{s2i} \) stress integral functions for the vertical stress in an oblique local co-ordinate system due to a linear variation of skin friction
\( P \) force for an end bearing and a skin friction pile
\( p_{s1i} - p_{s2i} \) stress elementary functions for the vertical stress in a global co-ordinate system due to a horizontal and vertical point load
\( p'_{s1i} - p'_{s2i} \) new stress elementary functions for the vertical stress in an oblique local co-ordinate system due to a horizontal and vertical point load
\( P_x \) a horizontal point load acting in the interior of the cross-anisotropic media
\( P_z \) a total load (force per unit length)
\( Q \) a vertical point load acting in the interior of the cross-anisotropic media
\( r_{s1i} - r_{s2i} \) stress integral functions for the vertical stress in an oblique local co-ordinate system due to a uniform skin friction
\( u_1, u_2 \) roots of the characteristic equation
\( X, Y, Z \) the global co-ordinate system
\( x, y, z \) the oblique local co-ordinate system

Greek letters
\( v \) Poisson’s ratio for the effect of horizontal stress on complementary horizontal strain
\( v' \) Poisson’s ratio for the effect of vertical stress on horizontal strain
\( \sigma'_{zz} \) vertical stress due to a linear variation of skin friction in an oblique local co-ordinate system
\( \sigma'_{zz} \) vertical stress due to a horizontal and vertical point load in a global co-ordinate system
\( \sigma'_{zz} \) vertical stress due to a horizontal and vertical point load in an oblique local co-ordinate system
\( \sigma'_{zz} \) vertical stress due to a uniform skin friction in an oblique local co-ordinate system

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