Wave propagation in an inhomogeneous cross-anisotropic medium

Cheng-Der Wang∗,†, Ya-Ting Lin, Yu-Shiuh Jeng and Zheng-Wei Ruan

Department of Civil and Disaster Prevention Engineering, National United University, No. 1, Lien Da, Kung-Ching Li, Miao-Li, Taiwan 360, Taiwan

SUMMARY

Analytical solutions for wave velocities and wave vectors are yielded for a continuously inhomogeneous cross-anisotropic medium, in which Young’s moduli \((E, E')\) and shear modulus \((G')\) varied exponentially as depth increased. However, for the rest moduli in cross-anisotropic materials, \(v\) and \(v'\) remained constant regardless of depth. We assume that cross-anisotropy planes are parallel to the horizontal surface. The generalized Hooke’s law, strain–displacement relationships, and equilibrium equations are integrated to constitute governing equations. In these equations, displacement components are fundamental variables and, hence, the solutions of three quasi-wave velocities, \(V_P\), \(V_{SV}\), and \(V_{SH}\), and the wave vectors, \(\mathbf{\vec{l}}_P\), \(\mathbf{\vec{l}}_{SV}\), and \(\mathbf{\vec{l}}_{SH}\), can be generated for the inhomogeneous cross-anisotropic media. The proposed solutions and those obtained by Daley and Hron, and Levin correlate well with each other when the inhomogeneity parameter, \(k\), is 0. Additionally, parametric study results indicate that the magnitudes and directions of wave velocity are markedly affected by (1) the inhomogeneous parameter, \(k\); (2) the type and degree of geomaterial anisotropy \((E/E', G'/E', \text{and } v/v')\); and (3) the phase angle, \(\theta\). Consequently, one must consider the influence of inhomogeneous characteristic when investigating the behaviors of wave propagation in a cross-anisotropic medium. Copyright © 2009 John Wiley & Sons, Ltd.

KEY WORDS: analytical solutions; wave velocity; wave vector; moduli varied exponentially as depth increased; inhomogeneous; cross-anisotropic medium

INTRODUCTION

Elastic wave propagation in anisotropic media is of significant interest in geophysics and other such applied sciences as soil dynamics, earthquake engineering, and petroleum engineering [1].

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In addition, wave propagation in inhomogeneous media is a classical problem in theories of acoustic, electromagnetic, elastic and seismic waves [2]. Many naturally occurring soils, such as flocculated clays, varved silts or sands, are typically deposited via sedimentation over long periods. The effects of deposition, overburden, and desiccation can cause soil media to exhibit both anisotropic and inhomogeneous deformability. The mechanical responses of anisotropic materials with compositional spatial gradients are called anisotropic functionally graded materials (FGMs), the characteristics of which are also employed in applied mechanics. Hence, in this study, the analytical solutions of body-wave velocities and vectors for a continuously inhomogeneous cross-anisotropic medium with two Young’s moduli and shear modulus varying exponentially with depth ($E e^{-kz}$, $E' e^{-kz}$, and $G' e^{-kz}$) are derived.

In considering the effect of inhomogeneity on wave propagation in an isotropic or cross-anisotropic medium, Table I presents a detailed survey of existing analytical or numerical solutions for inhomogeneous isotropic material; the types of inhomogeneity, and analytical or numerical results are listed. Corresponding to the isotropic solutions, very few studies have focused on an inhomogeneous anisotropic medium. The lack of analytical/numerical solutions is due to the rationality of simulation models or mathematical difficulties. Table II summarizes available solutions for an inhomogeneous anisotropic solid. No solutions have been obtained for body-wave velocities and their associated vectors, which propagate in an inhomogeneous cross-anisotropic medium with Young’s and shear moduli varying exponentially with depth. The governing equations for the desired solutions can be constructed by combining the generalized Hooke’s law, strain–displacement relationships, and equilibrium equations. Then, as wave propagation is on the $x$–$z$ plane, the solutions of three quasi-wave velocities, $V_P$, $V_{SV}$, and $V_{SH}$, and wave vectors, $\vec{l}_P$, $\vec{l}_{SV}$, and $\vec{l}_{SH}$, can be generated for inhomogeneous cross-anisotropic media. The yielded solutions demonstrate that the velocities and vectors are markedly influenced by (1) the inhomogeneity parameter ($k$), (2) the type and degree of material anisotropy ($E/E'$, $G'/E'$, and $v/v'$), and (3) the phase angle ($\theta$). The proposed solutions are identical to those acquired by Daley and Hron [54], and Levin [55], which are also given in Appendix A, when the inhomogeneity parameter, $k$, is 0. A parametric study is employed to demonstrate the derived solutions and elucidate the effects of these factors on body-wave velocities. Analytical results reveal that the inhomogeneous characteristic should be considered for waves propagating in isotropic/cross-anisotropic soil masses.

**ANALYTICAL SOLUTIONS OF BODY-WAVE VELOCITIES FOR AN INHOMOGENEOUS CROSS-ANISOTROPIC MEDIUM**

The simplest anisotropic case with broad geophysical and geotechnical applicability has only one direction (usually, but not always, vertical), while the other two directions are equivalent. This anisotropic case is called cross-anisotropy or transversely isotropy [56]. Figure 1 shows a Cartesian co-ordinate system ($x$, $y$, and $z$) chosen such that the $z$-axis is normal to the free surface of an inhomogeneous cross-anisotropic material. The $x$–$y$ plane is the plane of cross-anisotropy. The anisotropic medium has inhomogeneous elastic properties assumed to vary from point to point within the medium [57]. The expression of the stress–strain relationship for a continuously
Table I. Existing analytical/numerical solutions of related wave propagation problem for an inhomogeneous isotropic medium.

<table>
<thead>
<tr>
<th>Types of inhomogeneity</th>
<th>Author</th>
<th>Analytical or numerical results</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G = G_0 \ e^{cz}$</td>
<td>Wilson [3]</td>
<td>The propagation of Love and Rayleigh waves in the medium</td>
</tr>
<tr>
<td>$V_S^2(x_2) = V_{S\infty}^2 + (V_{S0}^2 - V_{S\infty}^2) \ e^{-\alpha x_2}$, $\rho(x_2) = \rho_{\infty} + (\rho_0 - \rho_{\infty}) \ e^{-\beta x_2}$, $x_2 \in [0, \infty]$</td>
<td>Rao and Goda [4]</td>
<td>Integral expressions for the surface displacements due to periodic point loads are obtained</td>
</tr>
<tr>
<td>$G(z) = G_0 + (G_{\infty} - G_0)(1 - e^{cz})$, $0 &lt; G_0 \leq G_{\infty}$</td>
<td>Vrettos [5]</td>
<td>The propagation characteristics of harmonic, free $SV/P$ surface waves. Dispersion relations and displacement distributions with depth are presented for the particular wave modes</td>
</tr>
<tr>
<td>$G(z) = G_0 + (G_{\infty} - G_0)(1 - e^{-cz})$ and $G(z) = G_0(1 + bz)^{1/3}$</td>
<td>Vrettos [6]</td>
<td>Analytical solutions for the displacement field at the surface due to a time-harmonic vertical line load</td>
</tr>
<tr>
<td>$G(z) = G_0 + (G_{\infty} - G_0)(1 - e^{cz})$, $0 &lt; G_0 \leq G_{\infty}$</td>
<td>Vrettos [7]</td>
<td>The dispersion relations for phase and group velocity, as well as the displacement distributions with depth for the propagation of horizontally polarized shear surface waves $(SH)$ wave)</td>
</tr>
<tr>
<td>$G(z) = G_0 + (G_{\infty} - G_0)(1 - e^{cz})$, $0 &lt; G_0 \leq G_{\infty}$</td>
<td>Vrettos [8]</td>
<td>Displacements and stresses in the interior of the half-space due to a time-harmonic vertical point load</td>
</tr>
<tr>
<td>$G(z) = G_0 + (G_{\infty} - G_0)(1 - e^{cz})$, $0 &lt; G_0 \leq G_{\infty}$</td>
<td>Vrettos [9]</td>
<td>Analytical solutions for the displacement field due to a time-harmonic surface line load</td>
</tr>
<tr>
<td>$G(z) = G_0 + (G_{\infty} - G_0)(1 - e^{cz})$, $0 &lt; G_0 \leq G_{\infty}$</td>
<td>Vrettos [10]</td>
<td>The surface displacement due to a time-harmonic surface tangential line load</td>
</tr>
<tr>
<td>$G(z) = G_0 + (G_{\infty} - G_0)(1 - e^{cz})$ and $G(z) = G_0(1 + mz)$</td>
<td>Leung et al. [11]</td>
<td>Using the frequency domain BEM to study the vibration isolation of structures from ground-transmitted waves by open trenches</td>
</tr>
<tr>
<td>$G(z) = G_{\infty} g(\xi), \ g(\xi) = 1 + \sum_{j=1}^{N} B_j \xi^j, \ \xi = e^{cz}, \ \sum_{j=1}^{N} B_j = -(1 - G_0/G_{\infty})$</td>
<td>Muravskii [12]</td>
<td>Solution of the time-harmonic problem due to vertical and horizontal forces applied to the surface</td>
</tr>
<tr>
<td>$G = G_2 + G_1 z$</td>
<td>Stoneley [13]</td>
<td>Dealing with the transmission of Rayleigh waves in an incompressible medium</td>
</tr>
<tr>
<td>$\mu_0 = \mu(\xi_0), \ \rho_0 = \rho(\xi_0); \ \hat{\mu}_0 = \mu(\xi_0), \ \hat{\rho}_0 = \rho(\xi_0)$, $z_0$ and $r_0$ are chosen units of length</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$G = G_0 + mz$</td>
<td>Awojobi [16]</td>
<td>Rayleigh waves in a semi-infinite incompressible medium in which a crust of rigidity varying linearly with depth lies on top of a uniform elastic medium of great depth Consider the three-dimensional axially symmetric stress equations of motion for the nonhomogeneity depending on $z$ and $r$ Stress and displacement fields due to a vertical vibration of a rigid circular body</td>
</tr>
</tbody>
</table>
Table I. Continued.

<table>
<thead>
<tr>
<th>Types of inhomogeneity</th>
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<th>Analytical or numerical results</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G = G_0 + mz )</td>
<td>Awojobi [17]</td>
<td>Stress and displacement fields due to vertical vibration of a rigid circular body and vertical vibration or rocking of a long rectangular body</td>
</tr>
<tr>
<td>( G = G_0 + mz )</td>
<td>Awojobi [18]</td>
<td>Stress and displacement fields due to a torsional vibration of a rigid circular body</td>
</tr>
<tr>
<td>( \gamma = \gamma_w + \gamma', G = G_0 + G \gamma z )</td>
<td>Vardoulakis [19]</td>
<td>Dealing with the harmonic Rayleigh-type and transverse surface waves for an incompressible, submerged, dense sand</td>
</tr>
<tr>
<td>( G = G_0(1 + 0.5 \delta z), G = G_0 \sqrt{1 + \delta z} )</td>
<td>Vardoulakis [20]</td>
<td>The dispersion of torsional surface waves in two types of inhomogeneity is discussed</td>
</tr>
<tr>
<td>( G = G_0(1 + mz) ) and ( Vardoulakis ) and ( Vrettos ) [21]</td>
<td>An analytical–numerical examination of the behavior of harmonic Rayleigh-type waves</td>
<td></td>
</tr>
<tr>
<td>( G = G_0 + mz )</td>
<td>Muravskii [22]</td>
<td>Displacements due to a time-harmonic vertical or horizontal surface load</td>
</tr>
<tr>
<td>( G = G_0 + mz )</td>
<td>Muravskii [23]</td>
<td>The horizontal vibration of points at the surface excited by a surface time-harmonic horizontal force, and vertical vibration due to a vertical force</td>
</tr>
<tr>
<td>( \psi_p(z) = \psi_{p0} \left( 1 + \frac{2z}{\psi_{p0}} Z \right)^{-1/2} )</td>
<td>Ben-Menahem [24]</td>
<td>The wave equations are solved in cylindrical co-ordinates under suitable boundary conditions and integral representations are obtained for the displacements</td>
</tr>
<tr>
<td>( \psi_s(z) = \psi_{s0} \left( 1 + \frac{2z}{\psi_{s0}} Z \right)^{-1/2} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \mu = \mu^{' \left( 1 + \frac{z-h}{z} \right)^2} )</td>
<td>Hudson [25]</td>
<td>A model consists of a half-space in which the elastic parameters vary in a continuous or discontinuous way with depth</td>
</tr>
<tr>
<td>( G / G_0 = (z/z_0)^{\mu}, \rho / \rho_0 = (z/z_0)^{\rho-2} )</td>
<td>Hook [26]</td>
<td>A generalization of the method of separation of both the dependent and independent variables of the vector wave equation of elasticity</td>
</tr>
<tr>
<td>( \mu / \mu_0 = (z/z_0)^2, \rho / \rho_0 = (z/z_0)^{2-2} )</td>
<td>Hook [27]</td>
<td>Exact expressions are obtained for the ( P, SV ) and ( SH ) displacements generated by impulsive point sources buried in unbounded media</td>
</tr>
<tr>
<td>( \mu = \mu_0(1+(z/\lambda))^2 ) or ( \mu = \mu_0 e^{\alpha z} )</td>
<td>Deresiewicz [28]</td>
<td>The propagation of Love waves in a homogeneous elastic layer supported on an inhomogeneous half-space</td>
</tr>
<tr>
<td>( \rho = \rho_0(1+(z/\lambda))^2 ) or ( \rho = \rho_0 e^{\beta z} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \mu = \mu_0 \left[ \left( z_1 - z_2 \frac{r}{a} \right)^2 \right]^{2/3} )</td>
<td>Singh and Ben-Menahem [29]</td>
<td>Decoupling of the vector wave equation of elasticity for radially inhomogeneous isotropic media</td>
</tr>
<tr>
<td>( \rho = \rho_0 \left( \frac{\mu}{\mu_0} \right)^{2/3} )</td>
<td></td>
<td></td>
</tr>
</tbody>
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Table I. Continued.

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<th>Types of inhomogeneity</th>
<th>Author</th>
<th>Analytical or numerical results</th>
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<tbody>
<tr>
<td>( G = G_0(1 + \varepsilon z)^2 ), ( \rho = \rho_0(1 + \varepsilon z)^2 )</td>
<td>Chakrabarty and De [30]</td>
<td>The stresses and displacements due to a concentrated line load moving with supersonic speed</td>
</tr>
<tr>
<td>( G(z) = G_0(1 + bz)^2 )</td>
<td>Gazetas [31]</td>
<td>An analytical-numerical formulation is presented for dynamic displacements of strip foundations with Poisson’s ratio = 0.25</td>
</tr>
<tr>
<td>( \mu = \mu_0(r/a)^i ), ( \rho = \rho_0(r/a)^{i-2} )</td>
<td>Watanabe [32]</td>
<td>Transient response of an inhomogeneous elastic solid to an impulsive SH-source is considered</td>
</tr>
<tr>
<td>( \mu(y) = \mu_0 e^{2iy} ) and ( \mu(y) = \mu_0(\beta y + \delta)^2 )</td>
<td>Fazil and Murat [33]</td>
<td>The mixed boundary value problem for an inhomogeneous medium bonded to a rigid subspace</td>
</tr>
<tr>
<td>( k(z) = k_0(1 + A z)^{1/2} ), ( k_0 ) = the reference homogeneous medium’s wave number</td>
<td>Manolis and Bagtzoglou [34]</td>
<td>The phenomenon of acoustic or elastic wave propagation under time-harmonic conditions is used as the vehicle for both deterministic and random models</td>
</tr>
<tr>
<td>( G(z) = G_0(1 + bz)^2 )</td>
<td>Guzina and Pak [35]</td>
<td>A method of evaluation via asymptotic decomposition for the singular Green’s functions is presented to analyze the displacements due to a variety of time-harmonic ring and point sources</td>
</tr>
<tr>
<td>( \mu(z) = (\alpha z + b)^2 ), ( \rho(z)/\mu(z) = \rho_0/\mu_0 )</td>
<td>Manolis and Shaw [36]</td>
<td>A fundamental solution is derived for time-harmonic elastic waves originating from a point source and propagating in a three-dimensional unbounded heterogeneous medium with Poisson’s ratio = 0.25</td>
</tr>
<tr>
<td>( \mu(z) = (\alpha z + b)^2 ), ( \rho(z)/\mu(z) = \rho_0/\mu_0 )</td>
<td>Manolis et al. [37]</td>
<td>Development free-space fundamental solutions for an inhomogeneous material under time-harmonic conditions by means of conformal mapping methods</td>
</tr>
<tr>
<td>( \mu(z) = (\alpha z + b)^2 ), ( \rho(x)/\mu(x) = \rho_0/\mu_0 )</td>
<td>Manolis et al. [38]</td>
<td>Development of an efficient BIEM for numerical solution of 2D elastodynamic problems in cracked inhomogeneous media</td>
</tr>
<tr>
<td>( \mu = \mu_0(1 + (ky/h))^p ), ( \rho = \rho_0(1 + (ky/h))^{p-2} )</td>
<td>Watanabe and Payton [2]</td>
<td>Impulsive and time-harmonic Green’s functions are obtained for SH waves in an inhomogeneous elastic solid</td>
</tr>
<tr>
<td>( \rho/\rho_0 = (G/G_0) = A \cosh^2(x + \beta z) )</td>
<td>Pekeris [39]</td>
<td>A power series expansion in the wavelength for the Rayleigh wave problem</td>
</tr>
<tr>
<td>( \rho = \rho(x_2), \lambda = \lambda(x_2), G = G(x_2) )</td>
<td>Rao [40]</td>
<td>Employing the pure stress equations of motion for the plane-strain case</td>
</tr>
<tr>
<td>Velocity gradient linear or parabolic</td>
<td>Acharya [41]</td>
<td>Integral expressions for vertical and horizontal displacement due to a point source are obtained in terms of reflection coefficients for P and S wave velocities increase monotonically with depth</td>
</tr>
</tbody>
</table>
Table II. Existing analytical/numerical solutions of related wave propagation problem for an inhomogeneous anisotropic medium.

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<th>Author</th>
<th>Analytical or numerical results</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G(z) = G_0 \left(1 + \gamma z^2\right)$, $\gamma = 0, 1, 2$</td>
<td>Waas et al. [42]</td>
<td>Solutions for the displacement caused by dynamic loads in a viscoelastic transversely isotropic medium</td>
</tr>
<tr>
<td>$A(z) = x_0 + \beta_0 z + \sum_{j=1}^{M} x_j \exp(-\gamma_j z)$</td>
<td>Shaw and Makris [43]</td>
<td>Fundamental Green’s functions are developed for the Helmholtz and Laplace equations</td>
</tr>
<tr>
<td>$G = G_0(1+\beta z)^m$, $\rho = \rho_0$</td>
<td>Dey et al. [44]</td>
<td>The propagation of Rayleigh waves in an inhomogeneous incompressible layer over a homogeneous incompressible half-space</td>
</tr>
<tr>
<td>$G = G_0(1+a z)^m$, $\rho = \rho_0(1+a z)^m$</td>
<td>Dey et al. [45]</td>
<td>The propagation of torsional surface waves in three types of nonhomogeneous media</td>
</tr>
<tr>
<td>$G = G_0 e^{iz}$, $\rho = \rho_0 e^{iz}$</td>
<td>Yanson [46]</td>
<td>The asymptotics of high-frequency Love waves of $\text{SH}$ type is considered for a transversely isotropic medium</td>
</tr>
<tr>
<td>$N = N_0 \cos \frac{\beta}{b}$, $L = L_0 \cos \frac{z}{b}$, $P = P_0 \cos \frac{\gamma}{b}$, $G = G_0 \cos \frac{z}{b}$, $\rho = \rho_0 \cos \frac{\beta}{b}$</td>
<td>Dey et al. [47]</td>
<td>The propagation of torsional surface waves under compressive initial stress in two types of nonhomogeneous media</td>
</tr>
<tr>
<td>$C_{44}(y, z) = C_{44}^0 h(y, z)$, $C_{66}(y, z) = C_{66}^0 h(y, z)$, $\rho(y, z) = \rho_0^0 h(y, z)$</td>
<td>Daros [48]</td>
<td>Pure stress equations of motion are used to model Bleustein–Gulyaev waves in an inhomogeneous, piezoelectric half-space</td>
</tr>
<tr>
<td>$C_{ijkl}(x) = C_{ijkl}^0 h(x)$, $\rho(x) = \rho_0 h(x)$</td>
<td>Rangelov et al. [49]</td>
<td>The combined algebraic plus Radon transforms are applied to the governing time-harmonic equations of motion for the inhomogeneous general anisotropic, material under plane-strain conditions to recover certain classes of fundamental solutions</td>
</tr>
<tr>
<td>$\hat{E}(\vec{z}) = m \frac{E_0(z)}{L_0} \hat{r}^2$, $E(r) = E_0 \left(\frac{L}{r}\right)^2 \hat{r}$, $\hat{p}(\vec{z}) = \rho_0 \left(\frac{L}{L_0}\right)^\beta + \rho_0 \left(\frac{L}{L_0}\right)^\beta + \rho_0 \left(\frac{L}{L_0}\right)^\beta + \rho_0 \left(\frac{L}{L_0}\right)^\beta$</td>
<td>Bazyar and Song [50]</td>
<td>The scale boundary finite element method is extended to simulate time-harmonic responses of transversely isotropic unbounded domains</td>
</tr>
</tbody>
</table>

where \( k \) is the inhomogeneity parameter; \( C_{ij} \) (i., j = 1–6) are the elastic moduli or elasticity constants of the medium, which can be presented in terms of the five independent elastic constants, \( E, E', v, v' \) and \( G' \) as

\[
C_{11} = \frac{E \left(1 - \frac{E'}{E} v'^2\right)}{(1 + v) \left(1 - v - \frac{2E}{E'} v'^2\right)}, \quad C_{13} = \frac{E v'}{1 - v - \frac{2E}{E'} v'^2}, \quad C_{33} = \frac{E'(1 - v)}{1 - v - \frac{2E}{E'} v'^2}
\]

\[
C_{44} = G', \quad C_{66} = \frac{E}{2(1 + v)}
\]

where

- \( E \) and \( E' \) are Young’s moduli in the plane of cross-anisotropy and in a direction normal to the cross-anisotropic plane, respectively.
- \( v \) and \( v' \) are Poisson’s ratios characterizing the lateral strain response in the cross-anisotropy plane relative to a stress acting parallel or normal to the cross-anisotropic plane, respectively.
- \( G' \) is the shear modulus in planes normal to the cross-anisotropic plane.

Table II. Continued.

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<th>Types of inhomogeneity</th>
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<th>Analytical or numerical results</th>
</tr>
</thead>
<tbody>
<tr>
<td>( {c_{44}(x_2), e_{15}(x_2), \varepsilon_{11}(x_2), \rho(x_2)} = {c_0, e_0^{\beta}, \varepsilon_1^{\beta}, \rho^{\beta}} f(x_2) )</td>
<td>Collet et al. [51]</td>
<td>Considering the problem of a piezoelectric shear-horizontal surface wave which leaves the interface ((x_2 = 0)) free of mechanical tractions and vanishes as ( x_2 ) goes to infinity (the Bleustein–Gulyaev wave)</td>
</tr>
<tr>
<td>( B_{1,5}(z) = \bar{B}_{1,5}[1 + \sigma(z + H)], \rho_e(z) = \rho_e[1 + \beta(z + H)] )</td>
<td>Ke et al. [52]</td>
<td>The propagation of Love waves in an inhomogeneous transversely isotropic fluid-saturated porous-layered half-space with linearly varying properties</td>
</tr>
<tr>
<td>( C_{44}(y, z) = C_{44}^0 h(y, z), ) ( C_{66}(y, z) = C_{66}^0 h(y, z), ) ( \rho(y, z) = \rho_0 h(y, z) )</td>
<td>Daros [53]</td>
<td>A fundamental solution is obtained for ( SH ) waves in a class of inhomogeneous anisotropic media in terms of the fundamental solution of the homogeneous media via a transmutation formula</td>
</tr>
</tbody>
</table>

\[
\begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{zz} \\
\tau_{yz} \\
\tau_{xz} \\
\tau_{xy}
\end{bmatrix} =
\begin{bmatrix}
C_{11} & C_{11} - 2C_{66} & C_{13} & 0 & 0 & 0 \\
C_{11} - 2C_{66} & C_{11} & C_{13} & 0 & 0 & 0 \\
C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{44} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\varepsilon_{zz} \\
\gamma_{yz} \\
\gamma_{xz} \\
\gamma_{xy}
\end{bmatrix} e^{-kz}
\]

(1)
Table III. Differences between the homogeneous and inhomogeneous cross-anisotropic elastic constants.

<table>
<thead>
<tr>
<th>Homogeneous [58]</th>
<th>Inhomogeneous [59, 60]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$</td>
<td>$E e^{-kz}$</td>
</tr>
<tr>
<td>$E'$</td>
<td>$E' e^{-kz}$</td>
</tr>
<tr>
<td>$v$</td>
<td>$v$</td>
</tr>
<tr>
<td>$v'$</td>
<td>$v'$</td>
</tr>
<tr>
<td>$G'$</td>
<td>$G' e^{-kz}$</td>
</tr>
</tbody>
</table>

Table III lists the differences between homogeneous cross-anisotropic elastic constants [58] and inhomogeneous ones [59, 60]. Notably, for an inhomogeneous cross-anisotropic material, such as that in Equation (1), only three engineering elastic constants ($E$, $E'$, and $G'$) exponentially depend on the inhomogeneity parameter, $k$; however, the two Poisson’s ratios, $v$ and $v'$, remain constant. Additionally, based on the inhomogeneity parameter, $k$, the following situations exist:

- $k > 0$ indicates a hardened surface; $E$, $E'$, and $G'$ decrease as the depth increases.
- $k = 0$ is the conventional homogeneous condition.
- $k < 0$ indicates a soft surface; $E$, $E'$, and $G'$ increase as the depth increases.

The expressions of the strain–displacement relationship for a small strain in a Cartesian coordinate system are

\[
\varepsilon_{xx} = \frac{\partial u_x}{\partial x} \tag{3a}
\]
\[
\varepsilon_{yy} = \frac{\partial u_y}{\partial y} \tag{3b}
\]
\[
\varepsilon_{zz} = \frac{\partial u_z}{\partial z} \tag{3c}
\]
\[ \gamma_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \]  
\[ \gamma_{xz} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \]  
\[ \gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \]

where \( u_x, u_y, \) and \( u_z \) are displacements of a point on the axis of the Cartesian coordinate system.

The differential forms of equations of motion are

\[ \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = \rho \frac{\partial^2 u_x}{\partial t^2} \]  
\[ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = \rho \frac{\partial^2 u_y}{\partial t^2} \]  
\[ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \rho \frac{\partial^2 u_z}{\partial t^2} \]

where \( \rho \) is the medium density.

By combining the stress–strain (Equation (1)) and strain–displacement (Equations (3a)–(3f)) relationships and inputting these relationships into Equations (4a)–(4c), the linear equations of motion for an elastic, inhomogeneous, cross-anisotropic medium in terms of displacement vector components can be regrouped as

\[ \left( C_{11} \frac{\partial^2}{\partial x^2} + C_{66} \frac{\partial^2}{\partial y^2} + C_{44} \frac{\partial^2}{\partial z^2} - kC_{44} \frac{\partial}{\partial z} \right) u_x + \left( C_{11} - C_{66} \right) \frac{\partial^2 u_y}{\partial x \partial y} \]

\[ + \left[ \left( C_{13} + C_{44} \right) \frac{\partial^2}{\partial x \partial z} - kC_{44} \frac{\partial}{\partial x} \right] u_z = \rho \frac{\partial^2 u_x}{\partial t^2} \]  
\[ \left( C_{11} - C_{66} \right) \frac{\partial^2 u_x}{\partial x \partial y} + \left( C_{66} \frac{\partial^2}{\partial x^2} + C_{11} \frac{\partial^2}{\partial y^2} + C_{44} \frac{\partial^2}{\partial z^2} - kC_{44} \frac{\partial}{\partial z} \right) u_y \]

\[ + \left[ \left( C_{13} + C_{44} \right) \frac{\partial^2}{\partial y \partial z} - kC_{44} \frac{\partial}{\partial y} \right] u_z = \rho \frac{\partial^2 u_y}{\partial t^2} \]  
\[ \left[ \left( C_{13} + C_{44} \right) \frac{\partial^2}{\partial x \partial z} - kC_{13} \frac{\partial}{\partial x} \right] u_x + \left[ \left( C_{13} + C_{44} \right) \frac{\partial^2}{\partial y \partial z} - kC_{13} \frac{\partial}{\partial y} \right] u_y \]

\[ + \left[ C_{44} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + C_{33} \frac{\partial^2}{\partial z^2} - kC_{33} \frac{\partial}{\partial z} \right] u_z = \rho \frac{\partial^2 u_z}{\partial t^2} \]
The direction of wave propagation is on the $x$–$z$ plane; thus, the displacement components for each particle can be assumed as [61]

$$u_x = A_1 f (x \sin \theta + z \cos \theta - V t)$$ \hspace{1cm} (6a)

$$u_y = A_2 f (x \sin \theta + z \cos \theta - V t)$$ \hspace{1cm} (6b)

$$u_z = A_3 f (x \sin \theta + z \cos \theta - V t)$$ \hspace{1cm} (6c)

where $A_1$–$A_3$ are constants that cannot be zero, $\theta$ is the phase angle between the normal wavefront and the unique axis ($z$-axis) (Figure 2), and $V$ is the desired body-wave velocity. Anisotropic media generally have three body waves propagating with velocities that vary as the direction of phase propagation varies. Their polarizations are orthogonal and fixed for a particular phase propagation direction. The waves are called quasi-waves as polarizations may not be along dynamic axes. Hence, replacing Equations (6a)–(6c) into Equations (5a)–(5c) generates the following governing equations:

$$\left( C_{11} \sin^2 \theta + C_{44} \cos^2 \theta - k C_{44} \cos \theta \right) A_1 + \left( (C_{13} + C_{44}) \sin \theta \cos \theta - k C_{44} \sin \theta \right) A_3 = A_1 \rho V^2 \hspace{1cm} (7a)$$

$$\left( C_{66} \sin^2 \theta + C_{44} \cos^2 \theta - k C_{44} \cos \theta \right) A_2 = A_2 \rho V^2 \hspace{1cm} (7b)$$

$$\left( (C_{13} + C_{44}) \sin \theta \cos \theta - k C_{13} \sin \theta \right) A_1 + \left( C_{44} \sin^2 \theta + C_{33} \cos^2 \theta - k C_{33} \cos \theta \right) A_3 = A_3 \rho V^2 \hspace{1cm} (7c)$$

where Equations (7a)–(7c) can be rewritten in the matrix form as

$$\begin{bmatrix} d_{11} - \rho V^2 & d_{12} & d_{13} \\ d_{21} & d_{22} - \rho V^2 & d_{23} \\ d_{31} & d_{32} & d_{33} - \rho V^2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \hspace{1cm} (8)$$

in which

$$d_{11} = C_{11} \sin^2 \theta + C_{44} \cos^2 \theta - k C_{44} \cos \theta$$

$$d_{12} = 0$$

$$d_{13} = (C_{13} + C_{44}) \sin \theta \cos \theta - k C_{44} \sin \theta$$
\[d_{21} = 0\]
\[d_{22} = C_{66} \sin^2 \theta + C_{44} \cos^2 \theta - k C_{44} \cos \theta\]
\[d_{23} = 0\]
\[d_{31} = (C_{13} + C_{44}) \sin \theta \cos \theta - k C_{13} \sin \theta\]
\[d_{32} = 0\]
\[d_{33} = C_{44} \sin^2 \theta + C_{33} \cos^2 \theta - k C_{33} \cos \theta\]

In an inhomogeneous cross-anisotropic medium, the eigenvalue problem is modeled by Equation (8). This problem can be solved using Mathematica [62] to generate three independent solutions: one quasi-longitudinal, one transverse, and one quasi-transverse for each propagation direction. The three body-waves are polarized in the orthogonal direction. The exact transverse wave has a polarization vector with no component in the three directions. It is denoted by \(SH\) and the other vector is denoted by \(SV\). These waves, \(V_{SH}\), \(V_{SV}\), and \(V_{P}\), are defined by the magnitude of phase velocity of the three quasi-waves in the direction of the unit vector for an inhomogeneous cross-anisotropic medium as

\[V_{SH} = \sqrt{\frac{C_{66} \sin^2 \theta + C_{44} \cos \theta (\cos \theta - k)}{\rho}}\] (9)

\[V_{SV} = \frac{1}{\sqrt{2\rho}} \sqrt{B_1 - \sqrt{B_1^2 + 4B_2}}\] (10)

\[V_{P} = \frac{1}{\sqrt{2\rho}} \sqrt{B_1 + \sqrt{B_1^2 + 4B_2}}\] (11)

where

\[B_1 = C_{33} + C_{44} + \sin^2 \theta(C_{11} - C_{33}) - k \cos \theta(C_{33} + C_{44})\]

\[B_2 = -C_{33}C_{44} \cos^2 \theta(k - \cos \theta)^2 - C_{11}C_{44} \sin^4 \theta\]

\[+ \sin^2 \theta[k^2 C_{13} C_{44} + \cos \theta(\cos \theta - k)(C_{13}^2 - C_{11}C_{33} + 2C_{13}C_{44})]\]

**ANALYTICAL SOLUTIONS OF BODY-WAVE VECTORS FOR AN INHOMOGENEOUS CROSS-ANISOTROPIC MEDIUM**

To derive the analytical solutions of body-wave vectors for an inhomogeneous cross-anisotropic medium, one must identify the distinct between phase angle (\(\theta\)) and group angle (\(\phi\)), along which energy propagates. The wavefront is locally perpendicular to the propagation vector, \(\vec{i}\), since \(\vec{i}\) points in the direction of maximum rate of phase increase. The phase velocity is called wavefront
velocity because it measures the velocity of advance of the wavefront along \( l(k, \theta) \). However, the wavefront is not spherical, and Figure 2 clearly indicates that \( \theta \) differs from \( \phi \). Following Berryman [63], the relationship for each wave type can be stated in terms of the wave vector as follows:

\[
\bar{l} = \bar{l}_x + \bar{l}_z = l_x \bar{x} + l_z \bar{z}
\]

(12)

where components \( \bar{l}_x \) and \( \bar{l}_z \) are

\[
l_x = l(k, \theta) \sin \theta = \frac{\omega}{V(k, \theta)} \sin \theta
\]

and \( l_z = l(k, \theta) \cos \theta = \frac{\omega}{V(k, \theta)} \cos \theta \)

and the scalar length is

\[
l(k, \theta) = \sqrt{l_x^2 + l_z^2} = \frac{\omega}{V(k, \theta)}
\]

where \( \omega \) is the angular frequency.

Substituting Equations (9)–(11) into Equation (12) we can derive the three body-wave vectors.

In addition, the group velocity, \( \bar{V}_g \), is derived by

\[
\bar{V}_g = \frac{\partial(l_v)}{\partial l_x} x + \frac{\partial(l_v)}{\partial l_z} z
\]

(13)

where the partial derivatives are taken while the other component of \( \bar{l} \) remains constant. Because of the similarity in form between Equation (13) and the typical expression for group velocity in a dispersive medium, \( \bar{V}_g \) is also called ray velocity and \( \phi \) is the group angle. Based on Figure 2, \( \phi \) is defined as

\[
\tan \phi = \frac{\frac{\partial(l_v)}{\partial l_x}}{\frac{\partial(l_v)}{\partial l_z}} = \frac{V \sin \theta + \frac{dV}{d\theta} \cos \theta}{V \cos \theta - \frac{dV}{d\theta} \sin \theta} = \tan \theta + \frac{1}{\frac{dV}{d\theta} V} \frac{\frac{dV}{d\theta}}{\sin \theta}
\]

(14)

Namely, the relationships between the phase angle (\( \theta \)) and group angle (\( \phi \)) for an inhomogeneous cross-anisotropic medium can be obtained by applying the differentiation of Equations (9)–(11) with respect to \( \theta \) as

\[
\frac{d}{d\theta} V_{SH} = \frac{[C_{44}k + 2(-C_{44} + C_{66}) \cos \theta] \sin \theta}{2 \sqrt{\rho \sqrt{C_{44} \cos \theta(-k + \cos \theta) + C_{66} \sin^2 \theta}}}
\]

(15)

\[
\frac{d}{d\theta} V_{SV} = \frac{F_1 + F_2 - \frac{F_3 + 2(F_4 - F_5) \sin 2\theta - 3F_6 - F_7}{4 \sqrt{F_8^2 + 4[F_9 + (F_{10} + F_{11}) \sin^2 \theta - F_{12}]}}}{2 \sqrt{\rho} \left( \sqrt{F_8 - \sqrt{F_8^2 + 4[F_9 + (F_{10} + F_{11}) \sin^2 \theta - F_{12}]}} \right)}
\]

(16)
This section presents an example to investigate the three phase velocities \( V_{PH}, V_{SV}, \) and \( V_P \) varied with the phase angle, \( \theta \). Isotropic soil (Soil 1) and cross-anisotropic soils (Soils 2–4) are the geomaterials. The influence of the degree of soil anisotropy, specified by ratios \( E/E', \ G'/E', \) and \( v/v' \) on the phase velocities is analyzed. Table IV lists the elastic properties of Soils 1–4. The selected domains of anisotropic variation generally follow the suggestions of Gazetas [64], such

\[
\frac{d}{d\theta} V_P = \frac{F_1 + F_2 + \frac{F_3 + 2(F_4 - F_5)\sin 2\theta - 3F_6 - F_7}{4\sqrt{F_8 + 4[F_9 + (F_{10} + F_{11})\sin^2 \theta - F_{12}]}}}{2\sqrt{2}\sqrt{\rho}\left(\sqrt{F_8 + \sqrt{F_9^2 + 4[F_9 + (F_{10} + F_{11})\sin^2 \theta - F_{12}]}\right)}
\]

(17)

where

\[
F_1 = (C_{33} + C_{44})k \sin \theta
\]

\[
F_2 = (C_{11} - C_{33})\sin 2\theta
\]

\[
F_3 = [2C_{13}^2 - C_{11}C_{33} + 3C_{33}^2 + C_{44}(C_{11} + 4C_{13} - 5C_{33} + 4C_{44})]k \sin \theta
\]

\[
F_4 = (C_{11} - C_{33})(C_{11} + C_{33} - 2C_{44})
\]

\[
F_5 = [(C_{33} - C_{44})^2 - 4C_{13}C_{44}]k^2
\]

\[
F_6 = [2C_{13}^2 - C_{33}(C_{11} + C_{33}) + C_{44}(C_{11} + 4C_{13} + 3C_{33})]k \sin 3\theta
\]

\[
F_7 = (C_{11} + 2C_{13} + C_{33})(C_{11} - 2C_{13} + C_{33} - 4C_{44})\sin 4\theta
\]

\[
F_8 = C_{33} + C_{44} - (C_{33} + C_{44})k \cos \theta + (C_{11} - C_{33})\sin^2 \theta
\]

\[
F_9 = -C_{33}C_{44}(k - \cos \theta)^2 \cos^2 \theta
\]

\[
F_{10} = C_{13}C_{44}k^2
\]

\[
F_{11} = (C_{13}^2 - C_{11}C_{33} + 2C_{13}C_{44})\cos \theta (\cos \theta - k)
\]

\[
F_{12} = C_{11}C_{44}\sin^4 \theta
\]

The scalar magnitude of the group velocity, \( V(k, \phi) \), is denoted in terms of phase velocity, \( V(k, \theta) \), as

\[
V(k, \phi) = \sqrt{V^2(k, \theta) + \left(\frac{dV(k, \theta)}{d\theta}\right)^2}
\]

(18)

Additionally, when \( \theta = 0^\circ \) (vertical propagation) and \( \theta = 90^\circ \) (horizontal propagation), group velocity equals phase velocity. 

PARAMETRIC STUDY

This section presents an example to investigate the three phase velocities \( V_{PH}, V_{SV}, \) and \( V_P \) varied with the phase angle, \( \theta \). Isotropic soil (Soil 1) and cross-anisotropic soils (Soils 2–4) are the geomaterials. The influence of the degree of soil anisotropy, specified by ratios \( E/E', \ G'/E', \) and \( v/v' \) on the phase velocities is analyzed. Table IV lists the elastic properties of Soils 1–4. The selected domains of anisotropic variation generally follow the suggestions of Gazetas [64], such

that the range of the $E/E'$ ratio is 0.6–4 for clays and is as low as 0.2 for sands. Hence, the range of the $E/E'$ ratio is 1.35–2.37 and that of the $G'/E'$ ratio is 0.23–0.44 [65–68]; notably, we assume the $\nu'/\nu'$ ratio is 0.75–1.5. The values of $E$ and $\nu$ (Table IV) are 50 MPa [69] and 0.3 [70], respectively. However, the effect of the inhomogeneity parameter, $k$, is $-0.5$, $-0.3$, $-0.1$, 0 (homogeneous) to 0.1, 0.3, and 0.5. As mentioned, $k<0$ ($-0.5$, $-0.3$, $-0.1$) is a soft surface, whereas $E$, $E'$, and $G'$ increase as the depth of soil media increases. Conversely, $k>0$ (0.1, 0.3, 0.5) indicates a hardened surface, and $E$, $E'$, and $G'$ decrease as the depth of soil media decreases.

Based on the obtained solutions, a Mathematica program [62] is written to calculate phase and group velocities and associated vectors. In this example, Equations (9)–(11) determine the distributions of the non-dimensional phase velocities $V_{SH}(k>=0)/V_{SH}(k=0)$, $V_{SV}(k>=0)/V_{SV}(k=0)$, and $V_{P}(k>=0)/V_{P}(k=0)$ vs phase angle $\theta$ (range, $0–90^\circ$), for the isotropic Soil 1/cross-anisotropic Soils 2–4 (Figures 3–5). Figures 3(a)–(d) show the non-dimensional phase velocity $V_{SH}(k>=0)/V_{SH}(k=0)$ (where the numerator denotes $V_{SH}$ in the cases of $k>0$ ($k=0.1$, 0.3, 0.5), $k=0$ (homogeneous), and $k<0$ ($k=-0.1$, $-0.3$, $-0.5$); however, the denominator expresses $V_{SH}$ in the case of $k=0$) vs $\theta$, for Soils 1–4. When the stiffness of isotropic Soil 1 ($E/E'=1.0$, $G'/E'=0.385$, $\nu'/\nu'=1.0$) decreases as depth increases (i.e. $k=0.1$, 0.3, and 0.5), the non-dimensional ratio of $V_{SH}(k>0)/V_{SH}(k=0)$ is less than that of the homogeneous case (i.e. $V_{SH}(k=0)/V_{SH}(k=0)=1$) (Figure 3(a)). Nevertheless, this trend is reversed for the ratio $V_{SH}(k<0)/V_{SH}(k=0)$ when Soil 1 stiffness increases as the depth of soil medium increases (i.e. $k=-0.1$, $-0.3$, and $-0.5$). The degree and magnitude of the non-dimensional ratio for a hardened surface ($k>0$) are more obvious than those of a soft surface ($k<0$). In particular, when $k=0.5$ and $-0.5$, the phase velocity, $V_{SH}$, is significantly different between a homogeneous isotropic medium and an inhomogeneous one. In addition, all ratios ($V_{SH}(k>=0)/V_{SH}(k=0)$) for Soil 1 approach 1 when $V_{SH}$ is propagating from the vertical ($0^\circ$) to the horizontal ($90^\circ$). This reflects the fact that the inhomogeneity parameter, $k$, does have a marked impact on $V_{SH}$.

Figure 3(b) displays the distribution of the non-dimensional ratio $V_{SH}(k>=0)/V_{SH}(k=0)$ for the cross-anisotropic Soil 2 ($E/E'=2.37$, $G'/E'=0.385$, and $\nu'/\nu'=1.0$) vs $\theta$. According to this figure, the trend and magnitude of $V_{SH}(k>=0)/V_{SH}(k=0)$ for Soil 2 differ from those of Soil 1. Increasing $E/E'$ from 1 to 2.37 profoundly affects the phase velocity, $V_{SH}$, in the cases of $k>0$ and $k<0$. For geomaterials, when $E=50$ MPa, the increase in $E/E'$ (from 1 to 2.37) decreases Young’s modulus in the vertical direction. That is, such an anisotropic soil increases deformability in the vertical direction and hence, the magnitudes of $V_{SH}(k>=0)/V_{SH}(k=0)$ for Soil 2 (Figure 3(b)) are less than those of Soil 1 (Figure 3(a)). For the effect of $G'/E'$ on $V_{SH}(k>=0)/V_{SH}(k=0)$, Figure 3(c) plots the effect of $G'/E'$ on $V_{SH}(k>=0)/V_{SH}(k=0)$ as the value of $\theta$ varies for Soil 3 ($E/E'=1.0$, $G'/E'=0.23$, and $\nu'/\nu'=1.0$). Although the trend of Figure 3(c) resembles that of Figure 3(b), according to our results, decreasing $G'/E'$ from 0.385

### Table IV. Elastic properties for isotropic/cross-anisotropic soils ($E=50$ MPa, $\nu=0.3$).

<table>
<thead>
<tr>
<th>Soil type</th>
<th>$E/E'$</th>
<th>$G'/E'$</th>
<th>$\nu'/\nu'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Soil 1. Isotropy</td>
<td>1.0</td>
<td>0.385</td>
<td>1.0</td>
</tr>
<tr>
<td>Soil 2. Cross-anisotropy</td>
<td>2.37</td>
<td>0.385</td>
<td>1.0</td>
</tr>
<tr>
<td>Soil 3. Cross-anisotropy</td>
<td>1.0</td>
<td>0.23</td>
<td>1.0</td>
</tr>
<tr>
<td>Soil 4. Cross-anisotropy</td>
<td>1.0</td>
<td>0.385</td>
<td>1.5</td>
</tr>
</tbody>
</table>
to 0.23 significantly alters the distributions of $V_{SH}$. Decreasing $G'/E'$ from 0.385 to 0.23 indicates that when $E/E'=1.0$ (50 MPa), the shear modulus in the vertical plane $(G')$ declines. Restated, the increased deformability in the vertical direction decreases $V_{SH}$. Figure 3(d) shows the effect of $\nu/'$ on the non-dimensional ratio of $V_{SH}$ for Soil 4 ($E/E'=1.0$, $G'/E'=0.385$, and $\nu/'=1.5$). Calculation results reveal that when $\nu/'$ increases from 1 to 1.5, the trend in Figure 3(d) does not differ from that of isotropic Soil 1 (Figure 3(a)); nevertheless, the inhomogeneity parameter, $k$, also impacts the distribution of $V_{SH}(k>0)/V_{SH}(k=0)$.

Figures 4(a)–(d) present the estimation of the non-dimensional ratio of $V_{SV}$ ($V_{SV}(k>0)/V_{SV}(k=0)$) vs phase angle $\theta$ for Soils 1–4, respectively. The distribution of $V_{SV}(k<0)$ differs from that of $V_{SH}(k<0)/V_{SH}(k=0)$ (Figure 3(a)) for isotropic Soil 1 (Figure 4(a)). Notably, when $k=0.5$ and $-0.5$, $k=0.3$ and $-0.3$, and $k=0.1$ and $-0.1$, the values

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Soil 1 with \( E/E' = 1.0, G'/E' = 0.385, \nu = 1.0 \).

Soil 2 with \( E/E' = 2.37, G'/E' = 0.385, \nu = 1.0 \).

Soil 3 with \( E/E' = 1.0, G'/E' = 0.23, \nu = 1.0 \).

Soil 4 with \( E/E' = 1.0, G'/E' = 0.385, \nu = 1.5 \).

Figure 4. The distributions of the non-dimensional phase velocity \( V_{SV}(k=\theta)/V_{SV}(k=0) \) vs \( \theta \) for the isotropic Soil 1 and cross-anisotropic Soils 2–4: (a) Soil 1; (b) Soil 2; (c) Soil 3; and (d) Soil 4.

of \( V_{SV}(k=\theta)/V_{SV}(k=0) \), respectively, approach 0.926, 0.973, and 0.997. Additionally, \( V_{SV} \) increases as \( \theta \) increases when \( k>0 \) and decreases as \( \theta \) decreases when \( k<0 \). Figures 4(b)–(d) plot \( V_{SV}(k=\theta)/V_{SV}(k=0) \) vs \( \theta \) for the inhomogeneous cross-anisotropic Soils 2–4. The patterns of \( V_{SV}(k=\theta)/V_{SV}(k=0) \) differ markedly when \( E/E' \) is increased from 1 (Figure 4(a)) to 2.37 (Figure 4(b)), as \( G'/E' \) is decreased from 0.385 (Figure 4(a)) to 0.23 (Figure 4(c)), and \( \nu/\nu' \) is increased from 1 (Figure 4(a)) to 1.5 (Figure 4(d)). That indicates that the inhomogeneous parameter, \( k \), and the type and degree of soil anisotropy (Soils 2–4) strongly influence \( V_{SV} \) (Figure 4).

Figures 5(a)–(d) draw \( V_P(k=\theta)/V_P(k=0) \) vs \( \theta \) for Soils 1–4. The distribution of estimated results (Figure 5(a)) for Soil 1 resembles that of \( V_{SH}(k=\theta)/V_{SH}(k=0) \) for Soil 1 (Figure 3(a)), except for the cases when \( k=0.5 \) and \(-0.5 \). These very similar patterns also apply to Soil 2 for
Figure 5. The distributions of the normalized phase vector $V_P(k > = < 0)/V_P(k = 0)$ vs $\theta$ for the isotropic Soil 1 and cross-anisotropic Soils 2–4: (a) Soil 1; (b) Soil 2; (c) Soil 3; and (d) Soil 4.

Figures 5(b) and 3(b). However, an obvious difference is found in Soil 3 (Figures 5(c) and 3(c)) and Soil 4 (Figures 5(d) and 3(d)). In addition, the magnitudes of $V_P(k > = < 0)/V_P(k = 0)$ for Soil 3 (Figure 5(c)) are greater than those of Soil 1 (Figure 5(a)), at approximately $\theta = 50^\circ$ when $k = -0.3$ and $-0.5$, and at about $\theta = 40^\circ$ when $k = 0.3$ and 0.5. Similarly, when $v = 0.3$, the decrease in $v'$ from 0.3 (Figure 5(a)) to 0.2 (Figure 5(d)) decreases the values of $V_P(k > = < 0)/V_P(k = 0)$ around the aforementioned phase angle and in the cases of the inhomogeneity parameter.

Parametric analytical results indicate that the calculation obtained using the three quasi-body-wave solutions when $k=0$ is identical to those obtained by Daley and Hron [54] and Levin [55] when the medium is a homogeneous cross-anisotropic material. Moreover, the analytical results (Figures 3–5) demonstrate that the three non-dimensional body-wave velocities
(\(V_{SH}(k>0)/V_{SH}(k=0)\), \(V_{SV}(k>=0)/V_{SV}(k=0)\), and \(V_{P}(k>=0)/V_{P}(k=0)\) depend on the inhomogeneity parameter \(k\) \((k>0)\) and \((k<0)\), the type and degree of soil anisotropy \((E/E', G'/E',\) and \(v/v')\), and the phase angle \(\theta\). The normalized body-wave vectors \(\vec{l}_{SH}/\omega, \vec{l}_{SV}/\omega, \) and \(\vec{l}_{P}/\omega\) vs phase angle \(\theta\) and group angle \(\phi\), for Soils 1–4, can also be plotted using Equation (12), and Equations (14)–(17), respectively. The analytical results in terms of \(1/V_{SH}, 1/V_{SV}, \) and \(1/V_{P}\) are analogous to those of \(V_{SH}, V_{SV}, \) and \(V_{P}\) (Figures 3–5), except for directional arrows. Although vector results are not addressed in this parametric study, they would describe the importance of the inhomogeneous characteristic for wave propagation in a cross-anisotropic medium.

CONCLUSIONS

Analytical solutions for body-wave velocities and vectors of a continuously inhomogeneous cross-anisotropic material are derived. The inhomogeneity of the medium is described by exponentially varying Young’s moduli \((E\) and \(E')\) and shear modulus \((G')\) with depth. We assume that planes of cross-anisotropy are parallel to the horizontal surface. Utilizing displacement components as fundamental variables, the three quasi-body-wave velocities \((V_{P}, V_{SV}, \) and \(V_{SH}\)) can be yielded by solving the eigenvalue problems. Additionally, following Berryman’s approach \([63]\), the related wave vectors \((\vec{l}_{P}, l_{SV}\) and \(l_{SH}\)) can also be generated.

The closed-form solutions in this study are governed by (1) the inhomogeneous parameter, \(k\); (2) the type and degree of material anisotropy, \(E/E', G'/E',\) and \(v/v';\) and (3) the phase angle, \(\theta.\) The proposed solutions are compared with homogeneous cross-anisotropic solutions obtained by Daley and Hron \([54]\), and Levin \([55]\) when the inhomogeneity parameter, \(k\), is 0; all solutions are in good agreement. Furthermore, a parametric study shows the effects of the aforementioned factors on body-wave velocities. Analytical results indicate that the influence of inhomogeneous characteristic should be considered for determining wave propagation in a cross-anisotropic geomaterial.

APPENDIX A

According to Daley and Hron \([54]\), and Levin \([55]\), the closed-form solutions of three body-wave velocities for a homogeneous cross-anisotropic medium, which are denoted by \(V_{SH(hca)}, V_{SV(hca)},\) and \(V_{P(hca)}\), are presented as

\[
V_{SH(hca)}(\theta) = \sqrt{\frac{C_{66}\sin^2 \theta + C_{44}\cos^2 \theta}{\rho}}
\]

\[
V_{SV(hca)}(\theta) = \sqrt{\frac{1}{2\rho}[C_{33} + C_{44} + (C_{11} - C_{33})\sin^2 \theta - D(\theta)]}
\]

\[
V_{P(hca)}(\theta) = \sqrt{\frac{1}{2\rho}[C_{33} + C_{44} + (C_{11} - C_{33})\sin^2 \theta + D(\theta)]}
\]

where
\[
D(\theta) = \left\{ \left( C_{33} - C_{44} \right)^2 + 2 \sin^2 \theta \left[ (C_{13} + C_{44})^2 - (C_{33} - C_{44})(C_{11} + C_{33} - 2C_{44}) \right] \right. \\
\left. + \sin^4 \theta [(C_{11} + C_{33} - 2C_{44})^2 - 4(C_{13} + C_{44})^2] \right\}^{1/2}
\]

The proposed solutions for the three phase velocities (Equations (9)–(11)) are identical to the solutions of Daley and Hron [54], and Levin [55] (Equations (A1)–(A3)), when the inhomogeneity parameter \( k \) is 0.

APPENDIX B: NOMENCLATURE

\( C_{11}, C_{13}, C_{33}, C_{44}, C_{66} \) elastic moduli or elasticity constants
\( E \) Young’s modulus in the horizontal direction
\( E' \) Young’s modulus in the vertical direction
\( G' \) shear modulus in the vertical plane
\( k \) inhomogeneity parameter
\( \vec{l}_P, \vec{l}_{SH}, \vec{l}_{SV} \) three quasi-body-wave vectors for an inhomogeneous cross-anisotropic medium
\( V_P, V_{SH}, V_{SV} \) three quasi-body-wave velocities for an inhomogeneous cross-anisotropic medium
\( V_{P(hca)}, V_{SH(hca)}, V_{SV(hca)} \) three quasi-body-wave velocities for a homogeneous cross-anisotropic medium (Appendix A)
\( x, y, z \) Cartesian co-ordinate system

Greek letters
\( \theta \) phase angle
\( \nu \) Poisson’s ratio for the effect of horizontal stress on complementary horizontal strain
\( \nu' \) Poisson’s ratio for the effect of vertical stress on horizontal strain
\( \rho \) medium density
\( \phi \) group angle
\( \omega \) angular frequency

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